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Calculus I & II

250 problems on limits, derivatives, integrals, series and polar curves, with every step worked and the common mistakes named

Free sample

Chapter 13: Integration: substitution and parts, problems and worked solutions

First edition, November 2026 · full book: 250 problems in 25 chapters

About this sample

This is Chapter 13 of *Calculus I & II*, complete: its 10 problems, then the worked solution to each one and the mistakes that most often go wrong on them. The full workbook has 250 problems in 25 chapters, 2 of them written only for the workbook, in two volumes: *Problems* and *Worked Solutions*.

About the workbook

This workbook follows the single-variable calculus sequence taught in most first-year university courses and in AP Calculus AB and BC: limits and continuity, differentiation and its applications, integration techniques and applications, sequences and series, and parametric and polar curves.

Problems run from routine to exam-hard within each chapter. Each solution gives one line of work per step with the reason beside it, and each chapter ends with the specific errors that cost marks on its problems.

Chapters marked as workbook-only appear nowhere else. The rest also have free versions at eulerpad.app/practice; this edition collects them in teaching order, renumbers their cross-references, and adds a full answer key.

Every numerical claim in every solution is verified by a computer check (finite differences, simulation or exact arithmetic) before the book is built.

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CHAPTER 13

Integration: substitution and parts

Two techniques cover every integral on this chapter: substitution, which undoes the chain rule, and integration by parts, which undoes the product rule. The work goes wrong in a few places that recur: a dx replaced without the factor that du brings, u and dv chosen the wrong way round, a sign lost from du, limits left unchanged after a substitution, and a value at infinity asserted instead of computed. These ten problems run from single substitutions and single applications of parts to parts applied twice, a definite integral, an improper one, and the integral of the sigmoid.

BEFORE YOU START

- $\int f\,dx = F + C$ means $F' = f$: F is an antiderivative of f, and the constant C stands for any constant, since adding a constant does not change the derivative.
- Substitution: set $u = g(x)$, so $du = g'(x)\,dx$, and rewrite the whole integrand, dx included, in terms of u. In a definite integral, change the limits with it: $x = a$ becomes $u = g(a)$.
- Integration by parts: $\int u\,dv = uv - \int v\,du$. It is the product rule $(uv)' = u'v + uv'$ integrated and rearranged. For a definite integral, $\int_a^b u\,dv = \big[uv\big]_a^b - \int_a^b v\,du$, where $\big[F\big]_a^b = F(b) - F(a)$.
- u has a meaning local to each problem: in a substitution it is the new variable, in parts it is the factor to be differentiated, with dv the part to be integrated. No problem uses both at once.
- An improper integral is a limit: $\int_0^\infty f\,dx$ means $\lim_{b\to\infty} \int_0^b f\,dx$.
- ln is the natural log, defined for positive arguments; $\int \frac{du}{u} = \ln|u| + C$ on any interval where $u \neq 0$.
- Every antiderivative below can be checked by differentiating it — do that before looking at the solution.

13.1 ●○○

$$\int x e^{x^2}\,dx$$

13.2 ●○○

$$\int \frac{2x}{1 + x^2}\,dx$$

13.3 ●●○

$$\int x e^{x}\, dx$$

13.4 ●●○

$$\int \ln x\, dx$$

13.5 ●●○

$$\int x \cos x\, dx$$

13.6 ●●●

$$\int e^{x} \sin x\, dx$$

13.7 ●●○

$$\int_{0}^{1} \frac{dx}{1+x^{2}}$$

13.8 ●●●

$$\int \tan x\, dx$$

13.9 ●●●

$$\int_{0}^{\infty} x e^{-x}\, dx$$

13.10 ●●●

$\int \sigma(x)\,dx$ where $\sigma(x) = \dfrac{1}{1+e^{-x}}$, and then $\displaystyle\int_{-\infty}^{0} \sigma(x)\,dx$.

CHAPTER 13 · WORKED SOLUTIONS

Integration: substitution and parts

ANSWER KEY

13.1	$\frac{1}{2}e^{x^2} + C$
13.2	$\ln(1 + x^2) + C$
13.3	$(x - 1)e^x + C$
13.4	$x \ln x - x + C$
13.5	$x \sin x + \cos x + C$
13.6	$\frac{1}{2}e^x(\sin x - \cos x) + C$
13.7	$\frac{\pi}{4}$
13.8	$-\ln\lvert\cos x\rvert + C$
13.9	1
13.10	$\ln(1 + e^x) + C$; and $\int_{-\infty}^{0} \sigma = \ln 2 - \lim_{a \to -\infty} \ln(1 + e^a) = \ln 2$

Problem 13.1

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$$\int x e^{x^2}\, dx$$

1	$u = x^2$, $du = 2x\, dx$.	e^{x^2} is e^u with inner function x^2, and the integrand also contains x, which is half of that inner function's derivative $2x$.
2	$x\, dx = \frac{1}{2}\, du$.	Divide both sides of step 1 by 2, so that the $x\, dx$ in the integral can be replaced as a whole.
3	$\int x e^{x^2}\, dx = \int e^u \cdot \frac{1}{2}\, du = \frac{1}{2}e^u + C$.	Substitute steps 1 and 2; $(e^u)' = e^u$, and the constant $\frac{1}{2}$ comes out of the integral.
4	$\frac{1}{2}e^{x^2} + C$	Put $u = x^2$ back. Check: by the chain rule, $\frac{d}{dx}\frac{1}{2}e^{x^2} = \frac{1}{2}e^{x^2} \cdot 2x = xe^{x^2}$.

Problem 13.2

●○○

$$\int \frac{2x}{1 + x^2}\, dx$$

1	$u = 1 + x^2$, $du = 2x\, dx$.	The numerator $2x\, dx$ is exactly the derivative of the denominator, times dx.

2	$\displaystyle\int \frac{2x}{1+x^2}\,dx = \int \frac{du}{u} = \ln\lvert u\rvert + C.$	The denominator becomes u and the numerator with dx becomes du.
3	$\ln(1+x^2) + C$	Put $u = 1 + x^2$ back; $1 + x^2 \geq 1 > 0$ for every x, so the absolute value can be dropped and the answer holds on the whole real line.

Problem 13.3

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$$\int xe^x\,dx$$

1	$u = x,\ dv = e^x\,dx;\ du = dx,\ v = e^x.$	Differentiating x makes it simpler, the constant 1, and e^x is its own antiderivative, so the new integral $\int v\,du$ has no factor of x left.
2	$\displaystyle\int xe^x\,dx = xe^x - \int e^x\,dx.$	Parts: $uv - \int v\,du$.
3	$xe^x - e^x + C.$	$\int e^x\,dx = e^x + C$.
4	$(x-1)e^x + C$	Factor out e^x. Check: by the product rule, $\big((x-1)e^x\big)' = e^x + (x-1)e^x = xe^x$.

Problem 13.4

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$$\int \ln x\,dx$$

1	$u = \ln x,\ dv = dx;\ du = \dfrac{dx}{x},\ v = x.$	No antiderivative of $\ln x$ is known yet, but its derivative, $1/x$, is simple; the other factor is then 1, and dx integrates to x. $\ln x$ needs $x > 0$.
2	$\displaystyle\int \ln x\,dx = x\ln x - \int x\cdot\frac{1}{x}\,dx.$	Parts: $uv - \int v\,du$.
3	$x\ln x - \int 1\,dx = x\ln x - x + C.$	$x\cdot\frac{1}{x} = 1$ for $x > 0$, and $\int 1\,dx = x + C$.
4	$x\ln x - x + C$	Check: $(x\ln x - x)' = \ln x + x\cdot\frac{1}{x} - 1 = \ln x$.

Problem 13.5

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$$\int x\cos x\,dx$$

1	$u = x,\ dv = \cos x\,dx;\ du = dx,\ v = \sin x.$	Differentiating x removes it, and $\cos x$ integrates to $\sin x$ without getting harder.
2	$\displaystyle\int x\cos x\,dx = x\sin x - \int \sin x\,dx.$	Parts: $uv - \int v\,du$.

3	$\int \sin x \, dx = -\cos x + C.$	$(-\cos x)' = \sin x.$
4	$x \sin x + \cos x + C$	Subtracting $-\cos x$ adds $\cos x$. Check: $(x \sin x + \cos x)' = \sin x + x \cos x - \sin x = x \cos x$.

Problem 13.6

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> $\int e^x \sin x \, dx$

1	Call the integral $I = \int e^x \sin x \, dx$.	Neither factor gets simpler when differentiated, so parts will not finish the job; the plan is to apply parts twice until I reappears, then solve for it.
2	$u = \sin x$, $dv = e^x \, dx$; $du = \cos x \, dx$, $v = e^x$.	e^x integrates to itself, so v stays as simple as dv.
3	$I = e^x \sin x - \int e^x \cos x \, dx$.	Parts: $uv - \int v \, du$.
4	For $\int e^x \cos x \, dx$: $u = \cos x$, $dv = e^x \, dx$; $du = -\sin x \, dx$, $v = e^x$.	Keep the same choice, trig as u and the exponential as dv. Swapping it would integrate the $\cos x$ back to $\sin x$ and undo step 3, giving only $I = I$.
5	$\int e^x \cos x \, dx = e^x \cos x - \int e^x(-\sin x) \, dx = e^x \cos x + I$.	Parts again; the minus from du turns the new integral into $+I$.
6	$I = e^x \sin x - e^x \cos x - I$.	Substitute step 5 into step 3. The original integral now appears on both sides.
7	$2I = e^x(\sin x - \cos x)$.	Add I to both sides and factor out e^x. The constants of integration from the two applications of parts are gathered into one C in the final line.
8	$\frac{1}{2}e^x(\sin x - \cos x) + C$	Divide by 2. Check: $\left(\frac{1}{2}e^x(\sin x - \cos x)\right)' = \frac{1}{2}e^x(\sin x - \cos x) + \frac{1}{2}e^x(\cos x + \sin x) = e^x \sin x$.

Problem 13.7

●●○

> $\int_0^1 \dfrac{dx}{1 + x^2}$

1	$y = \arctan x$ means $\tan y = x$ with $-\frac{\pi}{2} < y < \frac{\pi}{2}$.	arctan is the inverse of tan on $(-\frac{\pi}{2}, \frac{\pi}{2})$, where $\cos y > 0$ and tan is increasing, so each x has exactly one y.

2	$\dfrac{1}{\cos^2 y}\,y' = 1.$	Differentiate $\tan y = x$ with respect to x: the chain rule gives the derivative of tan at y, which is $1/\cos^2 y$ (quotient rule on $\sin y/\cos y$), times y'.
3	$y' = \cos^2 y = \dfrac{1}{1+\tan^2 y} = \dfrac{1}{1+x^2}.$	$1+\tan^2 y = \dfrac{\cos^2 y + \sin^2 y}{\cos^2 y} = \dfrac{1}{\cos^2 y}$, and $\tan y = x$.
4	$\displaystyle\int_0^1 \frac{dx}{1+x^2} = \big[\arctan x\big]_0^1 = \arctan 1 - \arctan 0.$	Step 3 makes $\arctan x$ an antiderivative of the integrand, and a definite integral is the antiderivative's value at the upper limit minus its value at the lower one.
5	$\arctan 1 = \frac{\pi}{4}$ and $\arctan 0 = 0.$	$\tan\frac{\pi}{4} = 1$ and $\tan 0 = 0$, and both angles lie in $(-\frac{\pi}{2}, \frac{\pi}{2})$.
6	$\dfrac{\pi}{4}$	$\frac{\pi}{4} - 0$. The integrand lies between $\frac{1}{2}$ and 1 on $[0,1]$, and $\frac{\pi}{4} \approx 0.785$ is between them.

Problem 13.8

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$$\int \tan x\,dx$$

1	$\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$, on an interval where $\cos x \neq 0$.	$\tan x$ is defined only where $\cos x \neq 0$; written as a quotient, the numerator is, up to sign, the derivative of the denominator.
2	$u = \cos x$, $du = -\sin x\,dx$, so $\sin x\,dx = -du$.	$(\cos x)' = -\sin x$; the minus sign has to be carried into the integral.
3	$\displaystyle\int \frac{\sin x}{\cos x}\,dx = \int \frac{-du}{u} = -\ln\lvert u\rvert + C.$	$\int du/u = \ln\lvert u\rvert + C$ for $u \neq 0$; the absolute value is needed because $\cos x$ is negative on half of the intervals where $\tan x$ is defined.
4	$-\ln\lvert\cos x\rvert + C$	Put $u = \cos x$ back. Check: where $\cos x > 0$, the chain rule gives $\frac{d}{dx}\ln\cos x = \dfrac{-\sin x}{\cos x} = -\tan x$; where $\cos x < 0$, it gives $\frac{d}{dx}\ln(-\cos x) = \dfrac{\sin x}{-\cos x} = -\tan x$. Either way the minus sign in front makes the derivative $\tan x$.

Problem 13.9

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$$\int_0^\infty x e^{-x}\,dx$$

1	$\displaystyle\int_0^\infty x e^{-x}\,dx = \lim_{b\to\infty}\int_0^b x e^{-x}\,dx.$	The upper limit is not a number, so the integral is defined as the limit of integrals over $[0, b]$.

2	$u = x$, $dv = e^{-x}\,dx$; $du = dx$, $v = -e^{-x}$.	Differentiating x removes it; $(-e^{-x})' = e^{-x}$ by the chain rule.
3	$\int_0^b xe^{-x}\,dx = \big[-xe^{-x}\big]_0^b + \int_0^b e^{-x}\,dx$.	Parts on $[0, b]$: $\big[uv\big]_0^b - \int_0^b v\,du$, and $-v = e^{-x}$.
4	$\big[-xe^{-x}\big]_0^b = -be^{-b} - 0$ and $\int_0^b e^{-x}\,dx = \big[-e^{-x}\big]_0^b = 1 - e^{-b}$.	Evaluate each bracket at b and at 0; at $x = 0$ the first bracket is $0 \cdot e^0 = 0$.
5	$\int_0^b xe^{-x}\,dx = 1 - e^{-b} - be^{-b}$.	Add the two parts of step 4.
6	$0 < be^{-b} = \dfrac{b}{e^b} < \dfrac{b}{b^2/2} = \dfrac{2}{b}$ for $b > 0$, so $be^{-b} \to 0$.	$e^b = 1 + b + \frac{b^2}{2} + \cdots > \frac{b^2}{2}$ for $b > 0$, since every term of the series is positive; the bound $2/b$ goes to 0.
7	1	Take $b \to \infty$ in step 5: $e^{-b} \to 0$ and, by step 6, $be^{-b} \to 0$. This is the mean of the exponential distribution with rate 1.

Problem 13.10

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> $\int \sigma(x)\,dx$ where $\sigma(x) = \dfrac{1}{1 + e^{-x}}$, and then $\int_{-\infty}^{0} \sigma(x)\,dx$.

1	$\sigma(x) = \dfrac{e^x}{1 + e^x}$.	Multiply top and bottom by e^x, which is never 0, so that the numerator becomes the derivative of the denominator.
2	$u = 1 + e^x$, $du = e^x\,dx$.	The denominator's derivative, e^x, is the numerator.
3	$\int \dfrac{e^x}{1 + e^x}\,dx = \int \dfrac{du}{u} = \ln\lvert u\rvert + C$.	Substitute step 2.
4	$\int \sigma(x)\,dx = \ln(1 + e^x) + C$.	Put $u = 1 + e^x$ back; $1 + e^x > 1$, so the absolute value can be dropped. This is softplus, whose derivative is σ (Differentiation rules, Problem 13.4).
5	$\int_{-\infty}^{0} \sigma(x)\,dx = \lim_{a \to -\infty} \int_a^0 \sigma(x)\,dx = \lim_{a \to -\infty} \big(\ln 2 - \ln(1 + e^a)\big)$.	The lower limit is not a number, so the integral is a limit; step 4 evaluated at 0 gives $\ln 2$, since $1 + e^0 = 2$, and at a gives $\ln(1 + e^a)$.
6	$\ln(1 + e^a) \to \ln 1 = 0$ as $a \to -\infty$.	$e^a \to 0$, and ln is continuous at 1.
7	$\ln(1 + e^x) + C$; **and** $\int_{-\infty}^{0} \sigma = \ln 2 - \lim_{a \to -\infty} \ln(1 + e^a) = \ln 2$	Steps 4 to 6. The area is finite although σ never reaches 0, because $\sigma(x) < e^x$ decays exponentially as $x \to -\infty$.

Where this goes wrong

1. Substituting without changing dx

Setting $u = x^2$ turns e^{x^2} into e^u at a glance, and the rest of the integrand is easy to wave through.

1	$\int xe^{x^2}\,dx$ with $u = x^2$	Right so far: the substitution of Problem 13.1.
2	"e^{x^2} becomes e^u, and $x\,dx$ becomes du."	The shortcut that causes the mistake: $x\,dx$ treated as the whole of du.
3	$\int xe^{x^2}\,dx = \int e^u\,du = e^{x^2} + C$ ✗	$du = 2x\,dx$, so $x\,dx = \frac{1}{2}\,du$, and the factor $\frac{1}{2}$ is lost. Differentiating the answer shows it: $(e^{x^2})' = 2xe^{x^2}$, twice the integrand. The correct antiderivative is $\frac{1}{2}e^{x^2} + C$.

2. Parts with u and dv swapped

Parts works with either choice of u and dv in the sense that the formula stays true, so it is tempting to take the first factor that integrates easily.

1	$\int xe^x\,dx$ by parts	Right so far: a product of a polynomial and an exponential, the case parts is for.
2	"x integrates easily too, so either factor can be dv."	The reasoning that causes the mistake: choosing dv by what can be integrated, without asking what the new integral will look like.
3	$u = e^x$, $dv = x\,dx$ in $\int xe^x\,dx$ ✗	This gives $v = \frac{x^2}{2}$, $du = e^x\,dx$ and $\int xe^x\,dx = \frac{x^2}{2}e^x - \int \frac{x^2}{2}e^x\,dx$ — true, but the new integral is harder than the old one, which is the sign that the choice is wrong. Pick u to be what gets simpler when differentiated: $u = x$ gives $(x-1)e^x + C$ (Problem 13.3).

3. Sign error in ∫ tan

$\tan x = \sin x / \cos x$ looks like a derivative over a function, and $\int u'/u = \ln|u|$ is the pattern that comes to mind.

1	$\int \tan x\,dx = \int \dfrac{\sin x}{\cos x}\,dx$ with $u = \cos x$	Right so far: the substitution of Problem 13.8.
2	"The numerator is the derivative of the denominator, so the answer is ln of the denominator."	The pattern that causes the mistake: $\sin x$ taken as the derivative of $\cos x$.

3	$\int \tan x \, dx = \ln \lvert \cos x \rvert + C$ ✗	$du = -\sin x \, dx$ brings a minus: $\sin x \, dx = -du$, so the integral is $-\ln \lvert \cos x \rvert + C$. Differentiating this line shows it: $(\ln \lvert \cos x \rvert)' = -\tan x$.

4. Changing the variable but keeping the old limits

A substitution in an indefinite integral changes only the integrand, and the habit carries over to definite integrals.

1	$\int_0^1 \frac{2x}{1+x^2} \, dx$ with $u = 1 + x^2$, $du = 2x \, dx$	Right so far: the substitution of Problem 13.2.
2	"The integrand becomes du/u; the rest stays."	The habit that causes the mistake: the limits, which are values of x, left in place.
3	$\int_0^1 \frac{2x}{1+x^2} \, dx = \int_0^1 \frac{du}{u}$ ✗	The limits must be values of u: $u(0) = 1$ and $u(1) = 2$, so the integral is $\int_1^2 du/u = \ln 2$. With the old limits the integral diverges, because $1/u$ is not integrable near $u = 0$ — yet the original integrand is continuous and bounded on $[0, 1]$, so its integral is finite.

5. Evaluating ∞ · 0 by feel

In Problem 13.9 the boundary term at infinity is x, which grows, times e^{-x}, which shrinks, and it is tempting to call the product 0 and move on.

1	$\int_0^\infty x e^{-x} \, dx = \big[-x e^{-x} \big]_0^\infty + \int_0^\infty e^{-x} \, dx$	Right so far as a plan: parts with $u = x$, $dv = e^{-x} \, dx$, as in Problem 13.9.
2	"At infinity e^{-x} is 0, so $-xe^{-x}$ is 0 there."	The reasoning that causes the mistake: $\infty \cdot 0$ read as 0.
3	$\big[-x e^{-x} \big]_0^\infty = 0 - 0$ ✗	The value happens to be right here, because $be^{-b} \to 0$ (Problem 13.9, step 6), but the line asserts it without a limit, and $\infty \cdot 0$ is not a value: $b \cdot \frac{1}{b} \to 1$, while $b \cdot b^{-1/2} \to \infty$. The same shortcut on $\int_1^\infty x \cdot \frac{1}{2} x^{-3/2} \, dx$, by parts with $u = x$ and $v = -x^{-1/2}$, calls the boundary term $\big[-x \cdot x^{-1/2} \big]_1^\infty$ zero at infinity, but at b it is $-\sqrt{b} \to -\infty$, and that integral diverges. Writing $\lim_{b \to \infty}$ is where the answer and the convergence are decided.

THE FULL WORKBOOK

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