Volumes of revolution: disks, washers and shells

Ten volumes by slicing: disks under y = √x, the cone and sphere formulas derived as checks, a washer between two curves, disks in y about the y-axis, shells about the y-axis, one solid by shells and by washers about x = 2, a washer about y = −1, a region that is one shell integral but two washer integrals, and a solid of square cross-sections, with worked solutions and the mistakes of squaring a difference of radii, forgetting a shifted axis and dropping the π.

BEFORE YOU START

Spinning a flat region around a line sweeps out a solid, and its volume is an integral of slices, the same idea as an area between curves with one more dimension. Thin slices across the axis are disks or washers; thin cylinders around the axis are shells. Either way the volume is an integral of an area, and the work goes wrong in a few places: the radii subtracted before squaring, a radius measured from the x-axis when the solid spins around some other line, an integrand in one variable with limits in the other, and a forgotten π or 2π. These ten problems cover each of them.

- **Disks.** Rotating the region under $y = f(x) \geq 0$ on $[a, b]$ about the x-axis gives slices that are disks of radius $f(x)$, so $V = \displaystyle\int_a^b \pi f(x)^2 \, dx$.
- **Washers.** If the region lies between an outer radius $R(x)$ and an inner radius $r(x)$, measured from the axis, each slice is a disk with a hole: $V = \displaystyle\int_a^b \pi\big(R(x)^2 - r(x)^2\big) \, dx$. Square each radius, then subtract.
- **Shells.** Rotating a vertical strip at x, of height $h(x)$, about a vertical axis sweeps a thin cylinder of radius $\rho(x)$, the strip's distance from the axis, with volume $2\pi\rho(x)h(x) \, dx$. So $V = \displaystyle\int_a^b 2\pi\rho(x)h(x) \, dx$.
- Slices are perpendicular to the axis (disks and washers); strips are parallel to it (shells). For a vertical axis, disks and washers are integrated in y and shells in x; for a horizontal axis it is the other way round.
- A radius is a distance from the axis. About $x = 2$, a point at x to the left of the axis has radius $2 - x$; about $y = -1$, a point at height y above it has radius $y + 1$.
- **Known cross-sections.** If the slice of a solid at x has area $A(x)$, its volume is $\displaystyle\int_a^b A(x) \, dx$. Disks and washers are the case $A = \pi R^2$ and $A = \pi(R^2 - r^2)$.

Builds on: Area between curves, Integration: substitution and parts

Problems

1. The region under $y = \sqrt{x}$ from $x = 0$ to $x = 4$ is rotated about the x-axis. Find the volume. difficulty ·

2. Rotate the line $y = \dfrac{r}{h}\,x$, for $0 \le x \le h$, about the x-axis, and show that the cone it sweeps has volume $\frac{1}{3}\pi r^2 h$. difficulty ·

3. Rotate the upper half of the circle $x^2 + y^2 = R^2$ about the x-axis, and show that the sphere it sweeps has volume $\frac{4}{3}\pi R^3$. difficulty ··

4. The region between $y = x$ and $y = x^2$ for $0 \le x \le 1$ is rotated about the x-axis. Find the volume. difficulty ··

5. The region bounded by $y = x^2$, the line $y = 4$ and the y-axis, with $x \ge 0$, is rotated about the y-axis. Find the volume with disks. difficulty ··

6. The region between $y = x$ and $y = x^2$ for $0 \le x \le 1$ is rotated about the y-axis. Find the volume with shells. difficulty ··

7. The same region, between $y = x$ and $y = x^2$ for $0 \le x \le 1$, is rotated about the line $x = 2$. Find the volume with shells, then check it with washers. difficulty ···

8. The region between $y = x$ and $y = x^2$ for $0 \le x \le 1$ is rotated about the line $y = -1$. Find the volume. difficulty ···

9. The region bounded by $y = \sqrt{x}$, the line $y = x - 2$ and the x-axis is rotated about the x-axis. Find the volume with shells, and check it with washers. difficulty ···

10. A solid has as its base the region between $y = 1 - x^2$ and the x-axis. Every cross-section perpendicular to the x-axis is a square with one side in the base. Find the volume. difficulty ··

Answers

1. $8\pi \approx 25.13$
2. $\frac{1}{3}\pi r^2 h$
3. $\frac{4}{3}\pi R^3$
4. $\frac{2\pi}{15} \approx 0.419$
5. $8\pi \approx 25.13$
6. $2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6} \approx 0.524$
7. $\frac{\pi}{2} \approx 1.571$, both ways
8. $\frac{7\pi}{15} \approx 1.466$
9. $\frac{16\pi}{3} \approx 16.76$, both ways
10. $\frac{16}{15} \approx 1.067$

Worked solutions

Problem 1

> The region under $y = \sqrt{x}$ from $x = 0$ to $x = 4$ is rotated about the x-axis. Find the volume.

A slice at x is a disk of radius $\sqrt{x}$, area $\pi(\sqrt{x})^2 = \pi x$.	The region touches the axis, so there is no hole.
$V = \displaystyle\int_0^4 \pi x \, dx = \pi \left[\frac{x^2}{2} \right]_0^4$	Add the disks from $x = 0$ to $x = 4$.
$8\pi \approx 25.13$	$\pi \cdot \frac{16}{2}$. The solid is a paraboloid, and its volume is half that of the cylinder of radius 2 and length 4 around it, 16π.

Problem 2

> Rotate the line $y = \dfrac{r}{h}\,x$, for $0 \le x \le h$, about the x-axis, and show that the cone it sweeps has volume $\frac{1}{3}\pi r^2 h$.

The line goes from $(0, 0)$ to (h, r), so the solid is a cone with tip at the origin, height h and base radius r.	The radius grows linearly from 0 to r.
$V = \displaystyle\int_0^h \pi \left(\frac{r}{h}\,x \right)^2 dx = \frac{\pi r^2}{h^2} \int_0^h x^2 \, dx$	Disks of radius $\frac{r}{h}x$; the constants come out of the integral.
$\dfrac{\pi r^2}{h^2} \cdot \dfrac{h^3}{3}$	$\int_0^h x^2 \, dx = \frac{h^3}{3}$.
$\frac{1}{3}\pi r^2 h$	A third of the cylinder $\pi r^2 h$, the familiar formula.

Problem 3

> Rotate the upper half of the circle $x^2 + y^2 = R^2$ about the x-axis, and show that the sphere it sweeps has volume $\frac{4}{3}\pi R^3$.

The upper half circle is $y = \sqrt{R^2 - x^2}$ for $-R \le x \le R$.	Solve for $y \ge 0$.
$V = \displaystyle\int_{-R}^{R} \pi\left(R^2 - x^2\right)\,dx$	Disks of radius $\sqrt{R^2 - x^2}$; squaring removes the root.
$\pi\left[R^2 x - \frac{x^3}{3}\right]_{-R}^{R} = \pi\left(\frac{2}{3}R^3 - \left(-\frac{2}{3}R^3\right)\right)$	At R: $R^3 - \frac{R^3}{3} = \frac{2}{3}R^3$; at $-R$ it is the negative.
$\frac{4}{3}\pi R^3$	$\frac{4}{3} = \frac{2}{3} + \frac{2}{3}$. Rotating the half circle, not the whole circle, avoids counting every slice twice.

Problem 4

> The region between $y = x$ and $y = x^2$ for $0 \le x \le 1$ is rotated about the x-axis. Find the volume.

On $(0, 1)$, $x > x^2$, so the outer radius is $R = x$ and the inner radius is $r = x^2$.	The region does not touch the axis between its ends, so each slice is a washer.
$V = \displaystyle\int_{0}^{1} \pi\left(x^2 - x^4\right)\,dx$	$\pi(R^2 - r^2)$: square each radius first.
$\pi\left(\frac{1}{3} - \frac{1}{5}\right)$	$\int_0^1 x^2\,dx = \frac{1}{3}$ and $\int_0^1 x^4\,dx = \frac{1}{5}$.
$\frac{2\pi}{15} \approx 0.419$	The cone swept by the line, $\frac{\pi}{3}$, minus the solid swept by the parabola, $\frac{\pi}{5}$.

Problem 5

> The region bounded by $y = x^2$, the line $y = 4$ and the y-axis, with $x \geq 0$, is rotated about the y-axis. Find the volume with disks.

Slices perpendicular to the y-axis are horizontal, so integrate in y, from $y = 0$ to $y = 4$.	The axis is vertical.
At height y the region runs from the y-axis to $x = \sqrt{y}$, so the disk has radius $\sqrt{y}$.	Solve $y = x^2$ for $x \geq 0$.
$V = \displaystyle\int_0^4 \pi y \, dy = \pi \left[\frac{y^2}{2} \right]_0^4$	$\pi(\sqrt{y})^2 = \pi y$.
$8\pi \approx 25.13$	Shells give the same: $\int_0^2 2\pi x \, (4 - x^2) \, dx = 2\pi(8 - 4) = 8\pi$.

Problem 6

> The region between $y = x$ and $y = x^2$ for $0 \leq x \leq 1$ is rotated about the y-axis. Find the volume with shells.

A vertical strip at x runs from $y = x^2$ up to $y = x$, so its height is $h(x) = x - x^2$.	Top minus bottom, as for an area.
The strip is at distance $\rho(x) = x$ from the y-axis.	Shells about a vertical axis are integrated in x.
$V = \displaystyle\int_0^1 2\pi x \, (x - x^2) \, dx = 2\pi \int_0^1 (x^2 - x^3) \, dx$	$2\pi \rho h \, dx$ for each shell.
$2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6} \approx 0.524$	Washers in y agree: the outer radius is $\sqrt{y}$ and the inner y, and $\int_0^1 \pi(y - y^2) \, dy = \frac{\pi}{6}$.

Problem 7

> The same region, between $y = x$ and $y = x^2$ for $0 \le x \le 1$, is rotated about the line $x = 2$. Find the volume with shells, then check it with washers.

Shells: the strip at x has height $x - x^2$ and is $2 - x$ from the axis.	The region lies to the left of $x = 2$, so the distance is $2 - x$, not x.
$V = \displaystyle\int_0^1 2\pi(2 - x)(x - x^2)\,dx =$ $2\pi \displaystyle\int_0^1 (2x - 3x^2 + x^3)\,dx = 2\pi\left(1 - 1 + \tfrac{1}{4}\right) = \tfrac{\pi}{2}$	$(2 - x)(x - x^2) = 2x - 2x^2 - x^2 + x^3$.
Washers: at height y the region runs from $x = y$ to $x = \sqrt{y}$. The far edge $x = y$ is $2 - y$ from the axis and the near edge $x = \sqrt{y}$ is $2 - \sqrt{y}$.	Horizontal slices perpendicular to the vertical axis; outer radius from the far edge, inner from the near one.
$\displaystyle\int_0^1 \pi\left((2 - y)^2 - (2 - \sqrt{y})^2\right) dy =$ $\pi \displaystyle\int_0^1 \left(4\sqrt{y} - 5y + y^2\right) dy = \pi\left(\tfrac{8}{3} - \tfrac{5}{2} + \tfrac{1}{3}\right)$	$(4 - 4y + y^2) - (4 - 4\sqrt{y} + y)$.
$\tfrac{\pi}{2} \approx 1.571$, **both ways**	$\tfrac{8}{3} + \tfrac{1}{3} = 3$ and $3 - \tfrac{5}{2} = \tfrac{1}{2}$. The solid is three times the one about the y-axis (Problem 6), because the region is farther from this axis.

Problem 8

> The region between $y = x$ and $y = x^2$ for $0 \le x \le 1$ is rotated about the line $y = -1$. Find the volume.

The axis is horizontal, so use washers in x. Every point of the region is above $y = -1$, so a point at height y is $y + 1$ from the axis.	A radius is a distance from the axis, not a y-coordinate.

$R = x + 1$ and $r = x^2 + 1$.	The line is the far edge and the parabola the near one.
$V = \pi \int_0^1 \left((x+1)^2 - (x^2+1)^2 \right) dx = \pi \int_0^1 \left(2x - x^2 - x^4 \right) dx$	$(x^2 + 2x + 1) - (x^4 + 2x^2 + 1)$.
$\pi \left(1 - \frac{1}{3} - \frac{1}{5} \right)$	Integrate term by term.
$\frac{7\pi}{15} \approx 1.466$	$\frac{15-5-3}{15}$. More than three times the volume about the x-axis (Problem 4), $\frac{2\pi}{15}$.

Problem 9

> The region bounded by $y = \sqrt{x}$, the line $y = x - 2$ and the x-axis is rotated about the x-axis. Find the volume with shells, and check it with washers.

In y, the region runs from $x = y^2$ on the left to $x = y + 2$ on the right, for $0 \le y \le 2$.	The same region as the area-between-curves page's Problem 5: the curves meet at $(4, 2)$.
Shells about the x-axis are horizontal strips: length $(y + 2) - y^2$, distance y from the axis.	Strips parallel to the axis.
$V = \int_0^2 2\pi y \, (y + 2 - y^2) \, dy = 2\pi \left[\frac{y^3}{3} + y^2 - \frac{y^4}{4} \right]_0^2 = 2\pi \left(\frac{8}{3} + 4 - 4 \right)$	One integral of a polynomial.
Washers in x: on $[0, 2]$ the slice is a disk of radius $\sqrt{x}$; on $[2, 4]$ it is a washer with $R = \sqrt{x}$ and $r = x - 2$.	The bottom of the region changes at $x = 2$, where the line crosses the axis, so washers need two integrals.

$\int_0^2 \pi x\,dx + \int_2^4 \pi\big(x-(x-2)^2\big)\,dx = 2\pi + \pi\Big(6-\frac{8}{3}\Big) = 2\pi + \frac{10\pi}{3}$	$\int_2^4 x\,dx = 6$ and $\int_2^4 (x-2)^2\,dx = \frac{8}{3}$.
$\frac{16\pi}{3} \approx 16.76$, **both ways**	Pick the method that keeps the region one piece in the variable you integrate.

Problem 10

> A solid has as its base the region between $y = 1 - x^2$ and the x-axis. Every cross-section perpendicular to the x-axis is a square with one side in the base. Find the volume.

The base runs from $x = -1$ to $x = 1$, and at x it is $1 - x^2$ tall.	$1 - x^2 = 0$ at $x = \pm 1$.
The cross-section at x is a square of side $1 - x^2$, area $A(x) = (1 - x^2)^2$.	Nothing is rotated: the slice's area comes from the shape it is told to be.
$V = \int_{-1}^{1} \big(1 - 2x^2 + x^4\big)\,dx = 2\Big(1 - \frac{2}{3} + \frac{1}{5}\Big)$	The integrand is even, so double the integral from 0 to 1.
$\frac{16}{15} \approx 1.067$	$2 \cdot \frac{8}{15}$. With semicircles of diameter $1 - x^2$ instead, $A = \frac{\pi}{8}(1 - x^2)^2$ and the volume is $\frac{\pi}{8} \cdot \frac{16}{15} = \frac{2\pi}{15}$.

Where this goes wrong

1. Squaring the difference of the radii

A washer is "the outer radius minus the inner radius", and the subtraction happens first.

$R = x$, $r = x^2$ on $[0, 1]$, about the x-axis	Right so far: Problem 4, step 1.
“The washer is $R - r$ thick, so its area is $\pi(R - r)^2$.”	The habit that causes the mistake: treating a washer as a disk whose radius is its thickness.

$V = \int_0^1 \pi(x - x^2)^2\,dx = \pi\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{\pi}{30} \approx 0.105$ ✗	That is the volume of a solid whose slices are disks as wide as the gap, which is not this solid. A washer is a big disk minus a small one, $\pi R^2 - \pi r^2$: the volume is $\frac{2\pi}{15} \approx 0.419$, four times as much.

2. Measuring the radius from the x-axis when the axis has moved

Most exercises rotate about the x-axis, where the radius is just y.

Region between $y = x$ and $y = x^2$, rotated about $y = -1$	Right so far: Problem 8.
"The outer radius is the top curve, x, and the inner is the bottom curve, x^2."	The habit that causes the mistake: reading the radii off the curves as if the axis were $y = 0$.
$V = \int_0^1 \pi\left(x^2 - x^4\right)\,dx = \frac{2\pi}{15} \approx 0.419$ ✗	That is the volume about the x-axis (Problem 4). About $y = -1$ every point is one unit farther from the axis, so $R = x + 1$ and $r = x^2 + 1$, and the volume is $\frac{7\pi}{15} \approx 1.466$. Draw the axis, then measure each radius from it.

3. Shells with radius x about x = 2

The shell formula is learned as $2\pi x\,h(x)$.

Height $x - x^2$ on $[0, 1]$, axis $x = 2$	Right so far: Problem 7, step 1.
"Shell volume is $2\pi x$ times the height."	The habit that causes the mistake: x is the radius only when the axis is the y-axis.

$V = \int_0^1 2\pi x\,(x - x^2)\,dx = \frac{\pi}{6} \approx 0.524$ ✗	That is the volume about the y-axis (Problem 6). The strip at x is $2 - x$ from the line $x = 2$, and with that radius the volume is $\frac{\pi}{2} \approx 1.571$. A quick check: the region is between 1 and 2 units from $x = 2$ but at most 1 unit from the y-axis, so the solid about $x = 2$ must be the bigger one.

4. Limits in one variable, integrand in the other

The region is described in x, and the limits get copied from that description.

Region under $y = 4$, right of the y-axis, left of $y = x^2$, about the y-axis; disk radius $\sqrt{y}$	Right so far: Problem 5, step 2.
"The region runs from $x = 0$ to $x = 2$, so those are the limits."	The habit that causes the mistake: the limits of x used for an integral in y.
$V = \int_0^2 \pi y\,dy = 2\pi \approx 6.28$ ✗	The disks stack along the y-axis, from $y = 0$ to $y = 4$, so the limits are heights. With them the volume is $8\pi \approx 25.13$. Whatever the variable after d, the limits and every radius must be written in it.

5. Dropping the 2π from a shell

Shells are often first met as "radius times height", with the constant left for later.

$\rho = x$, $h = x - x^2$ on $[0, 1]$, about the y-axis	Right so far: Problem 6, steps 1 and 2.
"Volume $= \int \rho\,h\,dx$."	The habit that causes the mistake: the circumference of the shell is $2\pi\rho$, not ρ.

$$V = \int_0^1 x\,(x - x^2)\,dx = \tfrac{1}{12} \approx 0.083$$ ✗

A shell unrolls to a thin slab $2\pi\rho$ long, h high and dx thick. Without the 2π the answer is off by a factor of about 6.28: the volume is $\frac{\pi}{6} \approx 0.524$. Every disk and washer volume has a π and every shell volume a 2π; a volume of revolution without one is a warning.

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