Vectors in 3-D: dot and cross products, lines and planes

Ten problems on vectors in three dimensions: lengths, the dot product and the angle between two vectors, work and the component along a direction, the cross product and the area of a parallelogram, a triangle's area and the plane through three points, the volume of a parallelepiped from a determinant, a line meeting a plane, the distance from a point to a plane, the angle and line of intersection of two planes, the distance between skew lines and a linear classifier's signed distances, with worked solutions and the mistakes of the middle sign of the cross product, dividing by squared lengths, forgetting to divide by |n|, a plane through the wrong point and the product of lengths as a volume.

BEFORE YOU START

Two products do most of the geometry in three dimensions. The dot product $u \cdot v$ measures how much two vectors point the same way: it gives lengths, angles, projections and the work done by a force, and a zero dot product means perpendicular. The cross product $u \times v$ builds a third vector perpendicular to both, whose length is the area of the parallelogram they span. A plane is described by a point and a normal vector, a line by a point and a direction, and every distance between them is a dot product with a unit vector. The same formulas reach machine learning unchanged: cosine similarity is the angle formula, and a linear classifier's decision boundary $w \cdot x + b = 0$ is a plane with normal w. These ten problems compute angles, areas and volumes, write lines and planes, find where they meet and how far apart they are, and end with the signed distance of a point to a decision boundary.

- For $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$: $u \cdot v = u_1v_1 + u_2v_2 + u_3v_3$ and $\|u\| = \sqrt{u \cdot u}$.
- **Angle.** $u \cdot v = \|u\|\,\|v\| \cos\theta$, so $\cos\theta = \dfrac{u \cdot v}{\|u\|\,\|v\|}$, with $0 \le \theta \le \pi$. $u \cdot v = 0$ exactly when $u \perp v$ (or one of them is 0).
- **Component and projection.** The component of v along u is $\dfrac{u \cdot v}{\|u\|}$ and the projection is $\dfrac{u \cdot v}{u \cdot u}\,u$.
- **Cross product.** $u \times v = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)$. It is perpendicular to u and v, follows the right-hand rule, $v \times u = -(u \times v)$, and $\|u \times v\| = \|u\|\,\|v\| \sin\theta$ is the area of the parallelogram on u and v.
- **Triple product.** $a \cdot (b \times c) = \det[a\ b\ c]$, the determinant with a, b, c as rows, and $|a \cdot (b \times c)|$ is the volume of the parallelepiped on a, b, c.

- **Lines and planes.** The line through P with direction d is $r(t) = P + td$. The plane through P_0 with normal $n = (a, b, c)$ is $n \cdot (x - P_0) = 0$, that is $ax + by + cz = n \cdot P_0$. The distance from a point Q to the plane $n \cdot x = k$ is $\dfrac{|n \cdot Q - k|}{\|n\|}$.

Problems

1. For $u = (1, 2, 2)$ and $v = (3, 0, 4)$, find $\|u\|$, $\|v\|$, $u \cdot v$ and the angle between them. difficulty ·

2. A force $F = (2, -1, 3)$ moves an object along the displacement $d = (4, 2, 1)$. Find the work done, the component of F along d and the projection of F onto d. difficulty ·

3. For $u = (1, 2, 2)$ and $v = (3, 0, 4)$, find $u \times v$, check that it is perpendicular to both, and find the area of the parallelogram and of the triangle on u and v. difficulty ·

4. Find the area of the triangle with vertices $A(1, 0, 0)$, $B(0, 2, 0)$, $C(0, 0, 3)$, and the equation of the plane through them. difficulty ··

5. Find the volume of the parallelepiped on $a = (1, 0, 1)$, $b = (2, 1, 0)$, $c = (0, 1, 3)$, and the volume of the tetrahedron with the same three edges from one vertex. difficulty ··

6. Find the line through $P(1, 2, 3)$ and $Q(4, 0, 5)$, and the point where it meets the plane $x + y + z = 12$. difficulty ··

7. Find the distance from $Q(2, 1, 3)$ to the plane $2x - y + 2z = 4$, and the point of the plane nearest to Q. difficulty ··

8. Find the angle between the planes $x + y + z = 1$ and $x - y + 2z = 3$, and their line of intersection. difficulty ··

9. Show that the lines $L_1 : (1, 0, 0) + s(1, 1, 0)$ and $L_2 : (0, 0, 2) + t(0, 1, 1)$ do not meet and are not parallel, and find the distance between them. difficulty ···

10. A linear classifier predicts the positive class when $w \cdot x + b > 0$, with $w = (2, -1, 2)$ and $b = -3$. Find the signed distances of $x_1 = (2, 0, 1)$ and $x_2 = (0, 1, 0)$ to the decision boundary, and the point of the boundary nearest to x_1. difficulty ···

Answers

1. $\|u\| = 3$, $\|v\| = 5$, $u \cdot v = 11$, $\theta \approx 42.83^\circ$
2. $W = 9$, component $\frac{9}{\sqrt{21}} \approx 1.964$, projection $\left(\frac{12}{7}, \frac{6}{7}, \frac{3}{7}\right)$
3. $u \times v = (8, 2, -6)$; parallelogram area $2\sqrt{26} \approx 10.20$, triangle area $\sqrt{26} \approx 5.10$
4. Area $\frac{7}{2}$; plane $6x + 3y + 2z = 6$
5. Parallelepiped 5, tetrahedron $\frac{5}{6}$
6. $r(t) = (1, 2, 3) + t(3, -2, 2)$; it meets the plane at $(7, -2, 7)$
7. Distance $\frac{5}{3}$; nearest point $\left(\frac{8}{9}, \frac{14}{9}, \frac{17}{9}\right)$
8. $\theta \approx 61.87^\circ$; line $(2, -1, 0) + t(3, -1, -2)$
9. Distance $\dfrac{1}{\sqrt{3}} \approx 0.5774$
10. Signed distances 1 and $-\frac{4}{3}$; nearest boundary point $\left(\frac{4}{3}, \frac{1}{3}, \frac{1}{3}\right)$

Worked solutions

Problem 1

> For $u = (1, 2, 2)$ and $v = (3, 0, 4)$, find $\|u\|$, $\|v\|$, $u \cdot v$ and the angle between them.

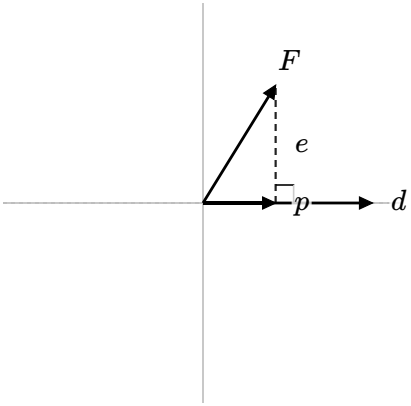
$\|u\| = \sqrt{1 + 4 + 4} = 3$ and $\|v\| = \sqrt{9 + 0 + 16} = 5$	Square, add, take the root.
$u \cdot v = 3 + 0 + 8 = 11$	Multiply matching entries and add.
$\cos\theta = \dfrac{11}{3 \cdot 5} = \dfrac{11}{15} \approx 0.7333$	Divide by the product of the lengths, not their squares.
$\theta = \arccos(0.7333) \approx 0.7476 \text{ rad}$	A positive dot product means an acute angle.
$\|u\| = 3$, $\|v\| = 5$, $u \cdot v = 11$, $\theta \approx 42.83^\circ$	In machine learning $\frac{11}{15}$ would be the cosine similarity of u and v. Dividing by $\|u\|^2\|v\|^2$ instead gives 87.2° (Mistake 2).

Problem 2

> A force $F = (2, -1, 3)$ moves an object along the displacement $d = (4, 2, 1)$. Find the work done, the component of F along d and the projection of F onto d.

$W = F \cdot d = 8 - 2 + 3 = 9$	Work is force times displacement, counting only the part of the force along the motion.
$\|d\| = \sqrt{16 + 4 + 1} = \sqrt{21} \approx 4.583$	Needed to turn d into a unit vector.
$\text{Component} = \dfrac{F \cdot d}{\|d\|} = \dfrac{9}{\sqrt{21}} \approx 1.964$	The dot product with the unit vector $d/\|d\|$.

Projection $= \dfrac{F \cdot d}{d \cdot d}\, d = \dfrac{9}{21}(4,2,1) = \dfrac{3}{7}(4,2,1)$	The component times the unit vector, which divides by $\|d\|$ twice. Drawn in the plane of F and d.



$W = 9$, **component** $\frac{9}{\sqrt{21}} \approx 1.964$, **projection** $\left(\frac{12}{7}, \frac{6}{7}, \frac{3}{7}\right)$	Check: F minus the projection is $\left(\frac{2}{7}, -\frac{13}{7}, \frac{18}{7}\right)$, and its dot product with d is $\frac{8-26+18}{7} = 0$.

Problem 3

For $u = (1,2,2)$ and $v = (3,0,4)$, find $u \times v$, check that it is perpendicular to both, and find the area of the parallelogram and of the triangle on u and v.

First entry: $u_2v_3 - u_3v_2 = 2 \cdot 4 - 2 \cdot 0 = 8$	Cover the first column and take the 2×2 determinant of what is left.
Second entry: $u_3v_1 - u_1v_3 = 2 \cdot 3 - 1 \cdot 4 = 2$; third: $u_1v_2 - u_2v_1 = 0 - 6 = -6$	The middle entry has the order reversed, which is the minus sign of the cofactor expansion (Mistake 1).
$(8,2,-6) \cdot (1,2,2) = 8 + 4 - 12 = 0$ and $(8,2,-6) \cdot (3,0,4) = 24 + 0 - 24 = 0$	The check catches almost every slip.

$$\underbrace{\begin{bmatrix} 1 & 2 & 2 \\ 3 & 0 & 4 \end{bmatrix}}_{\begin{bmatrix} u^\top \\ v^\top \end{bmatrix}} \cdot \underbrace{\begin{bmatrix} 8 \\ 2 \\ -6 \end{bmatrix}}_{u \times v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\|u \times v\| = \sqrt{64 + 4 + 36} = \sqrt{104} = 2\sqrt{26} \approx 10.198$	Check with Lagrange's identity: $\|u\|^2\|v\|^2 - (u \cdot v)^2 = 225 - 121 = 104$.
$u \times v = (8, 2, -6)$; **parallelogram area** $2\sqrt{26} \approx 10.20$, **triangle area** $\sqrt{26} \approx 5.10$	The triangle is half the parallelogram.

Problem 4

> Find the area of the triangle with vertices $A(1, 0, 0)$, $B(0, 2, 0)$, $C(0, 0, 3)$, and the equation of the plane through them.

$\vec{AB} = (-1, 2, 0)$ and $\vec{AC} = (-1, 0, 3)$	Two edges from the same vertex.
$n = \vec{AB} \times \vec{AC} = (2 \cdot 3 - 0 \cdot 0,\ 0 \cdot (-1) - (-1) \cdot 3,\ (-1) \cdot 0 - 2 \cdot (-1)) = (6, 3, 2)$	Perpendicular to both edges, so normal to the plane.
$\text{Area} = \frac{1}{2}\|n\| = \frac{1}{2}\sqrt{36 + 9 + 4} = \frac{7}{2}$	Half the parallelogram on the two edges.
$6x + 3y + 2z = n \cdot A = 6$	The constant comes from a point on the plane. Check B: 6; C: 6.
Area $\frac{7}{2}$; plane $6x + 3y + 2z = 6$	Dividing by 6 gives the intercept form $\frac{x}{1} + \frac{y}{2} + \frac{z}{3} = 1$. Taking the constant from the vector $\vec{AB}$ instead of a point gives a parallel plane through the origin (Mistake 4).

Problem 5

> Find the volume of the parallelepiped on $a = (1, 0, 1)$, $b = (2, 1, 0)$, $c = (0, 1, 3)$, and the volume of the tetrahedron with the same three edges from one vertex.

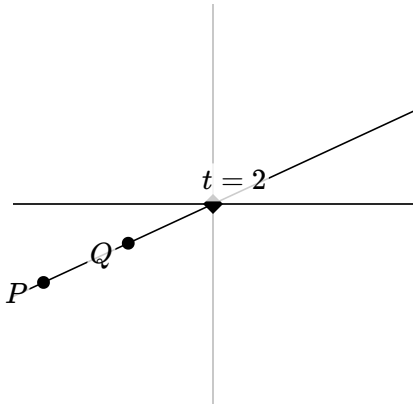
$\det \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = 1(1 \cdot 3 - 0 \cdot 1) - 0(2 \cdot 3 - 0 \cdot 0) + 1(2 \cdot 1 - 1 \cdot 0)$	Expand along the first row.

$= 3 - 0 + 2 = 5$	The triple product $a \cdot (b \times c)$.
Check: $b \times c = (3, -6, 2)$ and $a \cdot (3, -6, 2) = 3 + 0 + 2 = 5$	The same number computed the other way.
The tetrahedron is $\frac{1}{6}$ of the parallelepiped	Half for the triangular base, a third for the pyramid.
Parallelepiped 5, **tetrahedron** $\frac{5}{6}$	The sign of the determinant is positive, so a, b, c are a right-handed triple. The product of the lengths, $\sqrt{2} \cdot \sqrt{5} \cdot \sqrt{10} = 10$, is the volume only when the edges are perpendicular (Mistake 5).

Problem 6

> Find the line through $P(1, 2, 3)$ and $Q(4, 0, 5)$, and the point where it meets the plane $x + y + z = 12$.

$d = Q - P = (3, -2, 2)$	The direction is the difference of two points on the line.
$r(t) = (1 + 3t,\ 2 - 2t,\ 3 + 2t)$	$t = 0$ gives P and $t = 1$ gives Q.
$(1 + 3t) + (2 - 2t) + (3 + 2t) = 6 + 3t = 12$	Substitute the line into the plane: one equation in t.

$t = 2$ 	So the meeting point is beyond Q. Drawn in the plane that holds the line and the normal n, where $x + y + z = 12$ is the horizontal line.
$r(t) = (1, 2, 3) + t(3, -2, 2)$; **it meets the plane at** $(7, -2, 7)$	Check: $7 - 2 + 7 = 12$. If $n \cdot d$ had been 0 the line would be parallel to the plane, and the equation in step 3 would have no t in it.

Problem 7

> Find the distance from $Q(2, 1, 3)$ to the plane $2x - y + 2z = 4$, and the point of the plane nearest to Q.

$n = (2, -1, 2)$ and $\|n\| = 3$	The normal is read off the coefficients.
$n \cdot Q - 4 = 4 - 1 + 6 - 4 = 5$	How far the equation is from being satisfied, in units of $\|n\|$.
Distance $= \dfrac{\|5\|}{3} = \dfrac{5}{3}$	Dividing by $\|n\|$ turns it into a length.
Nearest point: $Q - \dfrac{5}{\|n\|^2}\, n = (2, 1, 3) - \dfrac{5}{9}(2, -1, 2) = \left(\dfrac{8}{9}, \dfrac{14}{9}, \dfrac{17}{9}\right)$	Move from Q along the normal by the right amount. Check: $\frac{16-14+34}{9} = 4$.
Distance $\frac{5}{3}$; nearest point $\left(\frac{8}{9}, \frac{14}{9}, \frac{17}{9}\right)$	The distance from Q to that point is $\frac{5}{9}\|n\| = \frac{5}{3}$. Forgetting to divide by $\|n\|$ gives 5 (Mistake 3).

Problem 8

> Find the angle between the planes $x + y + z = 1$ and $x - y + 2z = 3$, and their line of intersection.

$n_1 = (1, 1, 1)$, $n_2 = (1, -1, 2)$, $n_1 \cdot n_2 = 1 - 1 + 2 = 2$	The angle between two planes is the angle between their normals.
$\cos\theta = \dfrac{2}{\sqrt{3}\sqrt{6}} = \dfrac{\sqrt{2}}{3} \approx 0.4714$, so $\theta \approx 61.87^\circ$	$\sqrt{18} = 3\sqrt{2}$.
$d = n_1 \times n_2 = (1 \cdot 2 - 1 \cdot (-1),\ 1 \cdot 1 - 1 \cdot 2,\ 1 \cdot (-1) - 1 \cdot 1) = (3, -1, -2)$	The line lies in both planes, so it is perpendicular to both normals.
With $z = 0$: $x + y = 1$ and $x - y = 3$, so $(x, y) = (2, -1)$	Any point on both planes will do.
$\theta \approx 61.87^\circ$; **line** $(2, -1, 0) + t(3, -1, -2)$	Check a second point, $t = 1$: $(5, -2, -2)$ gives $5 - 2 - 2 = 1$ and $5 + 2 - 4 = 3$.

Problem 9

> Show that the lines $L_1 : (1, 0, 0) + s(1, 1, 0)$ and $L_2 : (0, 0, 2) + t(0, 1, 1)$ do not meet and are not parallel, and find the distance between them.

The directions $(1, 1, 0)$ and $(0, 1, 1)$ are not multiples of each other	So the lines are not parallel.
Meeting would need $1 + s = 0$, $s = t$, $0 = 2 + t$: $s = -1$ from the first, $t = -2$ from the third, and then $s \neq t$	The three equations have no common solution: the lines are skew.
$n = (1, 1, 0) \times (0, 1, 1) = (1, -1, 1)$	Perpendicular to both lines.

$P_2 - P_1 = (-1, 0, 2)$ and $(P_2 - P_1) \cdot n = -1 + 0 + 2 = 1$	Project the vector between any two points of the lines onto the common normal.
Distance $\dfrac{1}{\sqrt{3}} \approx 0.5774$	Divide by $\|n\| = \sqrt{3}$. The closest points are at $s = -\frac{4}{3}$ and $t = -\frac{5}{3}$: $\left(-\frac{1}{3}, -\frac{4}{3}, 0\right)$ on L_1 and $\left(0, -\frac{5}{3}, \frac{1}{3}\right)$ on L_2. Their difference is $\frac{1}{3}(1, -1, 1)$, a multiple of n, of length $\frac{1}{\sqrt{3}}$.

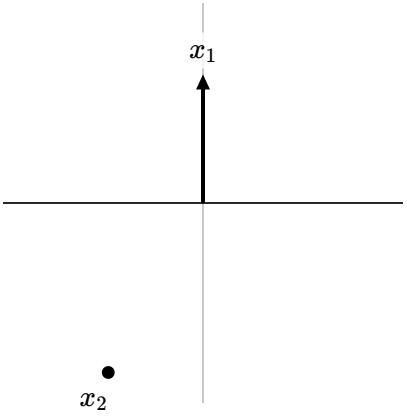
Problem 10

> A linear classifier predicts the positive class when $w \cdot x + b > 0$, with $w = (2, -1, 2)$ and $b = -3$. Find the signed distances of $x_1 = (2, 0, 1)$ and $x_2 = (0, 1, 0)$ to the decision boundary, and the point of the boundary nearest to x_1.

The boundary is the plane $w \cdot x = 3$, with normal w and $\|w\| = 3$	The weight vector is perpendicular to the decision boundary.
$w \cdot x_1 + b = 4 + 0 + 2 - 3 = 3$, so the signed distance is $\frac{3}{3} = 1$	Positive: x_1 is on the positive side.
$w \cdot x_2 + b = 0 - 1 + 0 - 3 = -4$, so the signed distance is $-\frac{4}{3}$	Negative: x_2 is classified negative, $\frac{4}{3}$ from the boundary.

Nearest point to x_1: $x_1 - 1 \cdot \dfrac{w}{\|w\|} = (2, 0, 1) - \dfrac{1}{3}(2, -1, 2) = \left(\dfrac{4}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right)$

Step back along the unit normal by the signed distance. Check: $\frac{8}{3} - \frac{1}{3} + \frac{2}{3} - 3 = 0$. Drawn in the plane through x_1 and x_2 that holds the normal: the boundary is the horizontal line, and the arrow is the unit normal from the nearest point up to x_1.



Signed distances 1 and $-\frac{4}{3}$; nearest boundary point $\left(\frac{4}{3}, \frac{1}{3}, \frac{1}{3}\right)$

The raw score $w \cdot x + b$ is the signed distance times $\|w\|$, so scaling w and b by 10 multiplies every score by 10 without moving the boundary. That is why a margin is measured as $\frac{w \cdot x + b}{\|w\|}$.

Where this goes wrong

1. The middle sign of the cross product

The three entries are computed with the same pattern, $u_i v_j - u_j v_i$ in index order.

$u = (1, 2, 2)$, $v = (3, 0, 4)$, first entry $u_2 v_3 - u_3 v_2 = 8$

Right so far: Problem 3.

“Second entry: $u_1 v_3 - u_3 v_1 = 4 - 6 = -2$. ”

The step that causes the mistake: the middle entry is $u_3 v_1 - u_1 v_3$, the minus sign of the cofactor expansion.

$u \times v = (8, -2, -6)$ ✗	The check catches it: $(8, -2, -6) \cdot (1, 2, 2) = 8 - 4 - 12 = -8 \neq 0$, so this vector is not perpendicular to u. The cross product is $(8, 2, -6)$.
$\underbrace{\begin{bmatrix} 1 & 2 & 2 \\ 3 & 0 & 4 \end{bmatrix}}_{\begin{bmatrix} u^\top \\ v^\top \end{bmatrix}} \cdot \begin{bmatrix} 8 \\ -2 \\ -6 \end{bmatrix} = \begin{bmatrix} -8 \\ 0 \end{bmatrix}$	

2. Dividing by the squared lengths

The angle formula is remembered with $u \cdot u$ and $v \cdot v$ in the denominator.

$u \cdot v = 11$, $\|u\|^2 = 9$, $\|v\|^2 = 25$	Right so far: Problem 1.
"$\cos\theta = \dfrac{u \cdot v}{\|u\|^2 \|v\|^2}$."	The squared lengths belong to the projection coefficient $\frac{u \cdot v}{u \cdot u}$, not to the cosine.
$\cos\theta = \frac{11}{225} \approx 0.0489$, so $\theta \approx 87.2^\circ$ ✗	Nearly perpendicular, for two vectors whose dot product is large and positive. A cosine must not change when u is doubled; this one halves. With $\|u\|\,\|v\| = 15$, $\theta \approx 42.83^\circ$.

3. The distance without dividing by |n|

The value of the plane's equation at the point is taken as the distance.

$n \cdot Q - 4 = 5$ for $Q = (2, 1, 3)$ and the plane $2x - y + 2z = 4$	Right so far: Problem 7.
"The distance is $\|n \cdot Q - 4\|$."	That number depends on how the equation is written: $4x - 2y + 4z = 8$ is the same plane and would give 10.

Distance $= 5$ ✗	5 is the distance measured in units of $\|n\| = 3$. Divide: the distance is $\frac{5}{3}$. Check with the nearest point $\left(\frac{8}{9}, \frac{14}{9}, \frac{17}{9}\right)$, which is $\frac{5}{3}$ from Q.

4. The plane through the wrong point

The constant on the right of the plane's equation is computed from an edge vector.

$n = (6, 3, 2)$ is normal to the plane through A, B, C	Right so far: Problem 4.
"$k = n \cdot \vec{AB} = -6 + 6 + 0 = 0$."	$\vec{AB}$ lies in the plane's direction but is not a point of it; n is perpendicular to every edge, so this always gives 0.
The plane is $6x + 3y + 2z = 0$ ✗	That plane passes through the origin, parallel to the right one. A does not satisfy it: $6 \neq 0$. Use a point: $k = n \cdot A = 6$.

5. The product of the lengths as a volume

The volume of a box is length times width times height, and the same is done for a slanted one.

$\|a\| = \sqrt{2}$, $\|b\| = \sqrt{5}$, $\|c\| = \sqrt{10}$	Right so far: the edges of Problem 5.
"Volume $= \|a\| \, \|b\| \, \|c\|$."	True only when the edges are perpendicular; here $a \cdot b = 2$ and $a \cdot c = 3$.
Volume $= \sqrt{2 \cdot 5 \cdot 10} = 10$ ✗	Twice the true volume. The product of the lengths is an upper bound (Hadamard's inequality), reached only by a rectangular box. The triple product gives 5.

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