# Triple integrals: iterated limits, cylindrical and spherical coordinates

Ten triple integrals worked by hand: a product over a box, the volume and centroid of a tetrahedron, limits read off a slanted plane, the solid under a paraboloid and its centroid in cylindrical coordinates, a moment of inertia, the volume of a ball in spherical coordinates, an ice-cream cone, the integral of ρ² over a ball, a change of order that makes an impossible integral easy, and the probability that a 3-D Gaussian lands in the unit ball, with worked solutions and the mistakes of dropping the r, leaving out sin φ, running φ to 2π, putting numbers where the inner limits need variables, and taking the cone's radius from the sphere.

**BEFORE YOU START**

A triple integral adds up a quantity over a solid: its volume when the integrand is 1, its mass when the integrand is a density, a probability when the integrand is a joint density of three variables. It is computed as three single integrals inside each other, and everything that goes wrong goes wrong in the limits or in the volume element. The inner limits may depend on the outer variables and the outer limits must be numbers. A round solid becomes a box in cylindrical or spherical coordinates, but only if the volume element changes with it: $dV = r\,dz\,dr\,d\theta$ in cylindrical and $dV = \rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta$ in spherical. These ten problems start with boxes and tetrahedra, move to paraboloids, cylinders, balls and cones, change the order of integration to rescue an integral that has no antiderivative, and end with the probability mass of a three-dimensional Gaussian inside the unit ball.

- **Iterated integrals.** Over a region $E$, $\iiint_E f\,dV$ is computed one variable at a time, innermost first. The innermost limits may depend on the two outer variables, the middle limits on the outer variable only, and the outer limits are numbers. To read off the limits, find where a line in the innermost direction enters and leaves $E$, then describe the shadow of $E$ on the remaining plane as in a double integral.
- **Volume and averages.** $\iiint_E 1\,dV$ is the volume of $E$. The centroid has $\bar{x} = \frac{1}{V}\iiint_E x\,dV$, and the same for $\bar{y}$, $\bar{z}$. A product $f(x)g(y)h(z)$ over a box splits into three single integrals.
- **Cylindrical coordinates.** $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, with $r \geq 0$ and $0 \leq \theta < 2\pi$. Then $x^2 + y^2 = r^2$ and $dV = r\,dz\,dr\,d\theta$: polar coordinates in the plane with $z$ unchanged.
- **Spherical coordinates.** $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, $z = \rho\cos\varphi$, where $\rho \geq 0$ is the distance from the origin, $\varphi \in [0, \pi]$ is the angle down from the positive $z$-axis and $\theta \in [0, 2\pi)$ is the polar angle in the $xy$-plane. Then $x^2 + y^2 + z^2 = \rho^2$ and $dV = \rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta$.

- The cone $z = \sqrt{x^2 + y^2}$ is $\varphi = \frac{\pi}{4}$ in spherical coordinates and $z = r$ in cylindrical. The standard normal CDF is $\Phi$, with $\Phi(1) = 0.8413$.

Builds on: Double integrals: iterated integrals, switching the order and polar coordinates

## Problems

**1.** Evaluate $\iiint_B xyz\,dV$ over the box $B = [0,1] \times [0,2] \times [0,3]$. difficulty ·

**2.** $T$ is the tetrahedron $x, y, z \geq 0$, $x + y + z \leq 1$. Find its volume and $\bar{x}$, the $x$-coordinate of its centroid. difficulty ·

**3.** Find the volume of the solid in the first octant under the plane $2x + 3y + z = 6$, writing the limits in the order $dz\,dy\,dx$. difficulty ··

**4.** $E$ is the solid between the paraboloid $z = x^2 + y^2$ and the plane $z = 4$. Find its volume and the height $\bar{z}$ of its centroid, in cylindrical coordinates. difficulty ··

**5.** A solid cylinder $x^2 + y^2 \leq 4$, $0 \leq z \leq 3$ has density $1$. Find its moment of inertia about the $z$-axis, $I_z = \iiint (x^2 + y^2)\,dV$. difficulty ·

**6.** Use spherical coordinates to show that a ball of radius $a$ has volume $\frac{4}{3}\pi a^3$. difficulty ·

**7.** Find the volume of the "ice-cream cone" inside the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $z = \sqrt{x^2 + y^2}$, first in spherical and then in cylindrical coordinates. difficulty ··

**8.** For the unit ball $B$, evaluate $\iiint_B (x^2 + y^2 + z^2)\,dV$, and use symmetry to get $\iiint_B z^2\,dV$. difficulty ··

**9.** Evaluate $\displaystyle\int_0^1 \int_z^1 \int_y^1 e^{x^3}\,dx\,dy\,dz$. difficulty ···

**10.** $X = (X_1, X_2, X_3)$ has independent standard normal coordinates, `difficulty ···`
with density $(2\pi)^{-3/2} e^{-\|x\|^2/2}$. Find the probability that $X$ lies in the unit ball.

## Answers

1. $\frac{9}{2} = 4.5$
2. $V = \frac{1}{6}$ and $\bar{x} = \frac{1}{4}$
3. $V = 6$
4. $V = 8\pi \approx 25.13$ and $\bar{z} = \dfrac{64\pi/3}{8\pi} = \dfrac{8}{3}$
5. $I_z = 24\pi \approx 75.40$
6. $V = \frac{4}{3}\pi a^3$
7. $V = \dfrac{8\pi(2-\sqrt{2})}{3} \approx 4.907$
8. $\iiint_B \rho^2 \, dV = \frac{4\pi}{5}$ and $\iiint_B z^2 \, dV = \frac{4\pi}{15}$
9. $\dfrac{e-1}{6} \approx 0.2864$
10. $P(\|X\| \le 1) \approx 0.1987$

## Worked solutions

### Problem 1

> Evaluate $\iiint_B xyz\,dV$ over the box $B = [0,1] \times [0,2] \times [0,3]$.

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|---|---|
| $\iiint_B xyz\,dV = \int_0^1 x\,dx \int_0^2 y\,dy \int_0^3 z\,dz$ | The integrand is a product of one function of each variable and the limits are all numbers, so the integral splits. |
| $\int_0^1 x\,dx = \frac{1}{2}$, $\int_0^2 y\,dy = 2$, $\int_0^3 z\,dz = \frac{9}{2}$ | Three single integrals. |
| $\frac{1}{2} \cdot 2 \cdot \frac{9}{2} = \frac{9}{2}$ | Multiply. |
| Iterated directly: $\int_0^3 z\,dz = \frac{9}{2}$, then $\int_0^2 \frac{9}{2} y\,dy = 9$, then $\int_0^1 9x\,dx = \frac{9}{2}$ | The same number, innermost first, as a check. |
| $\frac{9}{2} = 4.5$ | Splitting works only when both the integrand and the region are products; over a tetrahedron the limits tie the variables together. |

### Problem 2

> $T$ is the tetrahedron $x, y, z \geq 0$, $x + y + z \leq 1$. Find its volume and $\bar{x}$, the $x$-coordinate of its centroid.

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| $z$ runs from 0 up to $1 - x - y$; the shadow on the $xy$-plane is the triangle $x, y \geq 0$, $x + y \leq 1$ | A vertical line through $(x, y)$ enters $T$ at the floor and leaves through the slanted face. |
| $V = \int_0^1 \int_0^{1-x} (1 - x - y)\,dy\,dx = \int_0^1 \frac{(1-x)^2}{2}\,dx = \frac{1}{6}$ | The inner $y$-integral of $c - y$ from 0 to $c$, with $c = 1 - x$, is $\frac{c^2}{2}$. |

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| $\iiint_T x\,dV = \int_0^1 x\,\dfrac{(1-x)^2}{2}\,dx = \dfrac{1}{2}\left(\dfrac{1}{2} - \dfrac{2}{3} + \dfrac{1}{4}\right) = \dfrac{1}{24}$ | The same inner integrals, with the extra factor $x$ carried along. |
| $\bar{x} = \dfrac{1/24}{1/6} = \dfrac{1}{4}$ | Divide by the volume. |
| $V = \frac{1}{6}$ **and** $\bar{x} = \frac{1}{4}$ | By symmetry the centroid is $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$: a quarter of the way from the base to the opposite vertex, as for any tetrahedron. |

## Problem 3

> Find the volume of the solid in the first octant under the plane $2x + 3y + z = 6$, writing the limits in the order $dz\,dy\,dx$.

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| $0 \le z \le 6 - 2x - 3y$ | Floor $z = 0$, roof the plane solved for $z$. |
| $0 \le y \le \dfrac{6 - 2x}{3}$ | The shadow is where the roof is above the floor: $2x + 3y \le 6$ with $x, y \ge 0$. A line in the $y$-direction leaves it at $3y = 6 - 2x$. |
| $0 \le x \le 3$ | The shadow's extent in $x$, where $2x = 6$ on the $x$-axis. Only this outer range is a pair of numbers. |
| $\displaystyle\int_0^3 \int_0^{(6-2x)/3} (6 - 2x - 3y)\,dy\,dx = \int_0^3 \frac{(6-2x)^2}{6}\,dx = 6$ | With $c = 6 - 2x$, the inner integral of $c - 3y$ from 0 to $\frac{c}{3}$ is $\frac{c^2}{3} - \frac{c^2}{6} = \frac{c^2}{6}$; then $\int_0^3 \frac{(6-2x)^2}{6}\,dx = \frac{1}{6} \cdot \frac{216}{6} = 6$. |
| $V = 6$ | Check with the formula for a tetrahedron, $\frac{1}{6}$ of the product of the intercepts: $\frac{1}{6} \cdot 3 \cdot 2 \cdot 6 = 6$. Taking the limits as 0 to 6, 0 to 2 and 0 to 3 gives the box around it, 36 (Mistake 4). |

## Problem 4

> $E$ is the solid between the paraboloid $z = x^2 + y^2$ and the plane $z = 4$. Find its volume and the height $\bar{z}$ of its centroid, in cylindrical coordinates.

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| $r^2 \le z \le 4,\ 0 \le r \le 2,\ 0 \le \theta \le 2\pi$ | The paraboloid is $z = r^2$; it meets $z = 4$ at $r = 2$. |
| $V = \int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} r\,dz\,dr\,d\theta = 2\pi\int_0^2 (4 - r^2)\,r\,dr$ | The factor $r$ is part of $dV$; nothing in the integrand depends on $\theta$. |
| $2\pi\left[2r^2 - \dfrac{r^4}{4}\right]_0^2 = 2\pi(8 - 4) = 8\pi$ | Leaving out the $r$ gives $\frac{32\pi}{3}$ (Mistake 1). |
| $\iiint_E z\,dV = 2\pi\int_0^2 \dfrac{16 - r^4}{2}\,r\,dr = \pi\left(32 - \dfrac{32}{3}\right) = \dfrac{64\pi}{3}$ | The inner integral is $\int_{r^2}^{4} z\,dz = \frac{16 - r^4}{2}$. |
| $V = 8\pi \approx 25.13$ **and** $\bar{z} = \dfrac{64\pi/3}{8\pi} = \dfrac{8}{3}$ | The centroid is above the middle height 2 because the solid is wide at the top and narrows to a point at the bottom. |

## Problem 5

> A solid cylinder $x^2 + y^2 \le 4,\ 0 \le z \le 3$ has density 1. Find its moment of inertia about the $z$-axis, $I_z = \iiint (x^2 + y^2)\,dV$.

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| $x^2 + y^2 = r^2$ and $dV = r\,dz\,dr\,d\theta$, with $0 \le z \le 3,\ 0 \le r \le 2$ | In cylindrical coordinates the cylinder is a box. |
| $I_z = \int_0^{2\pi}\int_0^{2}\int_0^{3} r^2 \cdot r\,dz\,dr\,d\theta$ | The integrand $r^2$ times the $r$ from $dV$ gives $r^3$. |
| $= 2\pi \cdot \dfrac{2^4}{4} \cdot 3 = 24\pi$ | The integral splits: $\int_0^{2\pi} d\theta = 2\pi$, $\int_0^2 r^3\,dr = 4$, $\int_0^3 dz = 3$. |
| The mass is $M = \pi \cdot 2^2 \cdot 3 = 12\pi$, so $I_z = \frac{1}{2}MR^2$ with $R = 2$ | The textbook formula for a solid cylinder, as a check. |

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| $I_z = 24\pi \approx 75.40$ | Mass far from the axis counts with the square of its distance, which is why a hollow cylinder of the same mass has twice this moment. |

## Problem 6

> Use spherical coordinates to show that a ball of radius $a$ has volume $\frac{4}{3}\pi a^3$.

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| $0 \le \rho \le a,\ 0 \le \varphi \le \pi,\ 0 \le \theta \le 2\pi$ | $\varphi$ only needs to reach $\pi$, the negative $z$-axis; $\theta$ supplies the full turn. |
| $V = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$ | $dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$. |
| $\int_0^a \rho^2 \, d\rho = \dfrac{a^3}{3},\ \int_0^{\pi} \sin\varphi \, d\varphi = 2,\ \int_0^{2\pi} d\theta = 2\pi$ | The integrand is a product and the limits are numbers, so it splits. |
| $V = 2\pi \cdot 2 \cdot \dfrac{a^3}{3}$ | Multiply. |
| $V = \frac{4}{3}\pi a^3$ | Without the $\sin\varphi$ the result is $\frac{2}{3}\pi^2 a^3 \approx 6.58a^3$ (Mistake 2), and running $\varphi$ to $2\pi$ gives 0 (Mistake 3). |

## Problem 7

> Find the volume of the "ice-cream cone" inside the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $z = \sqrt{x^2 + y^2}$, first in spherical and then in cylindrical coordinates.

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| Spherical: $0 \le \rho \le 2,\ 0 \le \varphi \le \frac{\pi}{4},\ 0 \le \theta \le 2\pi$ | The cone makes the angle $\frac{\pi}{4}$ with the $z$-axis, and the solid is the part of the ball within that angle. |
| $V = 2\pi\Big[-\cos\varphi\Big]_0^{\pi/4} \cdot \dfrac{2^3}{3} = 2\pi\Big(1 - \dfrac{\sqrt{2}}{2}\Big)\dfrac{8}{3}$ | The integral splits as in Problem 6. |

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| Cylindrical: $r \le z \le \sqrt{4-r^2}$, and the cone meets the sphere where $r^2 + r^2 = 4$, at $r = \sqrt{2}$ | So $0 \le r \le \sqrt{2}$, not 2. |
| $2\pi \int_0^{\sqrt{2}} \left(\sqrt{4-r^2} - r\right) r \, dr = 2\pi\left(\dfrac{8-2\sqrt{2}}{3} - \dfrac{2\sqrt{2}}{3}\right)$ | $\int_0^{\sqrt{2}} r\sqrt{4-r^2}\,dr = \left[-\frac{1}{3}(4-r^2)^{3/2}\right]_0^{\sqrt{2}} = \frac{8-2\sqrt{2}}{3}$ and $\int_0^{\sqrt{2}} r^2\,dr = \frac{2\sqrt{2}}{3}$. |
| $V = \dfrac{8\pi(2-\sqrt{2})}{3} \approx 4.907$ | Both routes agree. Spherical coordinates turn the solid into a box, which is why the first route is one line. Running $r$ to 2 in the second route gives 0 (Mistake 5). |

## Problem 8

> For the unit ball $B$, evaluate $\iiint_B (x^2+y^2+z^2)\,dV$, and use symmetry to get $\iiint_B z^2\,dV$.

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| $x^2 + y^2 + z^2 = \rho^2$ | The integrand depends only on the distance from the origin. |
| $\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2 \cdot \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta = 2\pi \cdot 2 \cdot \dfrac{1}{5}$ | $\int_0^1 \rho^4\,d\rho = \frac{1}{5}$. |
| $\iiint_B (x^2+y^2+z^2)\,dV = \dfrac{4\pi}{5}$ | Multiply. |
| $\iiint_B x^2\,dV = \iiint_B y^2\,dV = \iiint_B z^2\,dV$ | The ball looks the same after relabeling the axes, so the three integrals are equal and add to $\frac{4\pi}{5}$. |
| $\iiint_B \rho^2\,dV = \frac{4\pi}{5}$ **and** $\iiint_B z^2\,dV = \frac{4\pi}{15}$ | Directly: $\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^4 \cos^2\varphi \sin\varphi \, d\rho\, d\varphi\, d\theta = 2\pi \cdot \frac{2}{3} \cdot \frac{1}{5} = \frac{4\pi}{15}$. Dividing by the volume $\frac{4\pi}{3}$, the average of $\rho^2$ over the ball is $\frac{3}{5}$. |

## Problem 9

> Evaluate $\int_0^1 \int_z^1 \int_y^1 e^{x^3}\,dx\,dy\,dz$.

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| As written the inner integral $\int_y^1 e^{x^3}\,dx$ has no elementary antiderivative | So change the order. |
| The region is $0 \le z \le y \le x \le 1$ | Read the inequalities off the limits: $z \le y$ from the middle, $y \le x$ from the inner, and all of them in $[0,1]$. |
| In the order $dz\,dy\,dx$: $0 \le z \le y$, $0 \le y \le x$, $0 \le x \le 1$ | For fixed $x$ and $y$, $z$ runs up to $y$; for fixed $x$, $y$ runs up to $x$. |
| $\int_0^1 \int_0^x \int_0^y e^{x^3}\,dz\,dy\,dx = \int_0^1 e^{x^3} \int_0^x y\,dy\,dx = \int_0^1 \frac{x^2}{2} e^{x^3}\,dx$ | $e^{x^3}$ is constant in the two inner integrals. |
| $\frac{e-1}{6} \approx 0.2864$ | With $u = x^3$, $\int_0^1 \frac{x^2}{2} e^{x^3}\,dx = \frac{1}{6}\int_0^1 e^u\,du$. The factor $\frac{x^2}{2}$ is the area of the triangle $0 \le z \le y \le x$ in the $yz$-plane, and it is exactly what the substitution needed. |

## Problem 10

> $X = (X_1, X_2, X_3)$ has independent standard normal coordinates, with density $(2\pi)^{-3/2} e^{-\|x\|^2/2}$. Find the probability that $X$ lies in the unit ball.

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| $P(\|X\| \le 1) = (2\pi)^{-3/2} \int_0^{2\pi}\int_0^{\pi}\int_0^1 e^{-\rho^2/2}\rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta$ | The density depends only on $\rho$, so spherical coordinates turn the ball into a box. |
| $= (2\pi)^{-3/2} \cdot 4\pi \int_0^1 \rho^2 e^{-\rho^2/2}\,d\rho = \sqrt{\frac{2}{\pi}} \int_0^1 \rho^2 e^{-\rho^2/2}\,d\rho$ | The angles give $4\pi$, the area of the unit sphere, and $4\pi(2\pi)^{-3/2} = \sqrt{2/\pi}$. |

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| $\int_0^1 \rho^2 e^{-\rho^2/2}\,d\rho = \Big[-\rho e^{-\rho^2/2}\Big]_0^1 + \int_0^1 e^{-\rho^2/2}\,d\rho = -e^{-1/2} + \sqrt{2\pi}\Big(\Phi(1) - \frac{1}{2}\Big)$ | Parts with $u = \rho$ and $dv = \rho e^{-\rho^2/2}\,d\rho$; the last integral is a normal probability. |
| $P = 2\Phi(1) - 1 - \sqrt{\frac{2}{\pi}}\,e^{-1/2} \approx 0.6827 - 0.4839$ | Multiply through by $\sqrt{2/\pi}$. |
| $P(\|X\| \le 1) \approx 0.1987$ | Each coordinate is within 1 of 0 with probability 0.6827, yet the vector is within distance 1 less than 20% of the time. In $d$ dimensions $\|X\|$ concentrates near $\sqrt{d}$, so a high-dimensional Gaussian has almost no mass near its mode. |

## Where this goes wrong

### 1. Dropping the r in cylindrical coordinates

The integrand is rewritten in $r$ and $\theta$, and $dV$ is taken as $dz\,dr\,d\theta$.

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| $r^2 \le z \le 4,\ 0 \le r \le 2,\ 0 \le \theta \le 2\pi$ | Right so far: Problem 4. |
| "$V = \int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} dz\,dr\,d\theta$." | The step that causes the mistake: a box $dz\,dr\,d\theta$ has volume $r\,dz\,dr\,d\theta$, because the arc it spans grows with $r$. |
| $V = 2\pi\int_0^2 (4 - r^2)\,dr = \frac{32\pi}{3} \approx 33.51$ ✗ | Compare the cylinder $r \le 2,\ 0 \le z \le 4$, of volume $16\pi \approx 50.27$: the paraboloid cuts away exactly half of it, so the answer is $8\pi \approx 25.13$. Without the $r$, a thin ring near the axis counts as much as a ring of the same width at the rim. |

### 2. Spherical coordinates without sin φ

The volume element is taken as $\rho^2\,d\rho\,d\varphi\,d\theta$.

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| $0 \le \rho \le a,\ 0 \le \varphi \le \pi,\ 0 \le \theta \le 2\pi$ | Right so far: Problem 6. |

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| "$dV = \rho^2\, d\rho\, d\varphi\, d\theta$." | The step that causes the mistake: a circle of latitude at angle $\varphi$ has radius $\rho \sin\varphi$, which shrinks to 0 at the poles, and the volume element has to shrink with it. |
| $V = 2\pi \cdot \pi \cdot \frac{a^3}{3} = \frac{2}{3}\pi^2 a^3 \approx 6.58a^3$ ✗ | The true volume is $\frac{4}{3}\pi a^3 \approx 4.19a^3$. Without $\sin\varphi$, the thin polar regions count as if they were as wide as the equator. |

### 3. φ running from 0 to 2π

Both angles are given the full turn.

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| $dV = \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$ and $0 \le \theta \le 2\pi$ | Right so far: Problem 6. |
| "$0 \le \varphi \le 2\pi$." | $\varphi$ is measured from the positive $z$-axis and reaches the negative $z$-axis at $\pi$; $\theta$ already sweeps all the way around. |
| $\int_0^{2\pi} \sin\varphi\, d\varphi = 0$, so $V = 0$ ✗ | A volume of 0 for a ball is a sure sign. For $\pi < \varphi < 2\pi$, $\sin\varphi < 0$ and the ball is counted a second time with a negative sign. With $0 \le \varphi \le \pi$, $V = \frac{4}{3}\pi a^3$. |

### 4. Numbers where the inner limits need variables

The limits are read off the intercepts, as if the solid were a box.

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| The plane $2x + 3y + z = 6$ meets the axes at $x = 3$, $y = 2$, $z = 6$ | Right so far: Problem 3. |
| "$0 \le z \le 6$, $0 \le y \le 2$, $0 \le x \le 3$." | Only the outer limits can be numbers; the roof $z = 6 - 2x - 3y$ comes down as $x$ and $y$ grow. |
| $V = 3 \cdot 2 \cdot 6 = 36$ ✗ | That is the box around the tetrahedron, six times its volume. With $z$ up to $6 - 2x - 3y$ and $y$ up to $\frac{6-2x}{3}$, $V = 6$. |

## 5. The cone's radius taken from the sphere

In cylindrical coordinates the outer radius is read off the sphere.

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| $r \le z \le \sqrt{4 - r^2}$ | Right so far: Problem 7. |
| “The sphere has radius 2, so $0 \le r \le 2$.” | The solid only reaches $r = \sqrt{2}$, where the cone meets the sphere. Beyond it the "floor" $z = r$ is above the "roof" $\sqrt{4 - r^2}$. |
| $V = 2\pi \int_0^2 \left(\sqrt{4 - r^2} - r\right) r \, dr = 2\pi\left(\frac{8}{3} - \frac{8}{3}\right) = 0$ ✗ | The negative part from $\sqrt{2}$ to 2 exactly cancels the solid. Set the floor equal to the roof to find the edge of the shadow: $r = \sqrt{4 - r^2}$ gives $r = \sqrt{2}$ and $V = \frac{8\pi(2-\sqrt{2})}{3} \approx 4.907$. |

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