Trigonometric substitution: sin, tan and sec, the triangle and the limits

Ten integrals by trigonometric substitution: an arcsine with new limits, the area formula for $\sqrt{(1 - x^2)}$, $x^3/\sqrt{(4 - x^2)}$ two ways, $1/(1 + x^2)^2$ by $x = \tan\theta$, $\ln(1 + \sqrt{2})$ from $\int \sec\theta$, a secant substitution read back off the triangle, completing the square first, the area of an ellipse, $\sqrt{(x^2 - 4)}/x$ on $[2, 4]$, and the normalizing constant of a Student-t distribution, with worked solutions and the mistakes of dropping the dx, keeping the x-limits and integrating $\tan^2\theta$ by the power rule.

BEFORE YOU START

A square root of a quadratic, $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$ or $\sqrt{x^2 - a^2}$, has no obvious antiderivative and nothing for an ordinary substitution to grab. Trigonometric substitution replaces x with a trig function chosen so that a Pythagorean identity turns the quadratic into a perfect square, and the root disappears. Circle and ellipse areas, arcsines and arctangents, and the normalizing constants of heavy-tailed distributions all come out this way. The work goes wrong in a few places: the dx left as $d\theta$, θ-integrals evaluated at x-limits, a square completed without balancing the constant, the wrong side read off the triangle, and $\tan^2\theta$ integrated as if it were a power of θ. These ten problems cover each of them.

- **Which substitution.** $\sqrt{a^2 - x^2}$: $x = a\sin\theta$, $dx = a\cos\theta\,d\theta$, and the root is $a\cos\theta$. $\sqrt{a^2 + x^2}$: $x = a\tan\theta$, $dx = a\sec^2\theta\,d\theta$, and the root is $a\sec\theta$. $\sqrt{x^2 - a^2}$: $x = a\sec\theta$, $dx = a\sec\theta\tan\theta\,d\theta$, and the root is $a\tan\theta$.
- **Why the root simplifies.** $1 - \sin^2\theta = \cos^2\theta$, $1 + \tan^2\theta = \sec^2\theta$ and $\sec^2\theta - 1 = \tan^2\theta$. The ranges $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ for sin, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ for tan and $0 \le \theta < \frac{\pi}{2}$ for sec (with $x \ge a$) keep the result positive, so no absolute value is needed.
- **Back to x.** Draw a right triangle from the substitution: $\sin\theta = \frac{x}{a}$ means opposite x, hypotenuse a, adjacent $\sqrt{a^2 - x^2}$. Every other trig function of θ is a ratio of two of its sides.
- **Definite integrals.** Change the limits to θ with the substitution and never return to x; or return to x first and use the original limits. Not a mixture.
- **Integrals you will need.** $\int \cos^2\theta\,d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4}$, $\int \sin^2\theta\,d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4}$, $\int \tan^2\theta\,d\theta = \tan\theta - \theta$ and $\int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|$.
- **Completing the square.** $x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + c - \frac{b^2}{4}$ turns any quadratic into one of the three forms.

Builds on: Integration: substitution and parts, Trig identities

Problems

1. Evaluate $\displaystyle\int_0^3 \frac{dx}{\sqrt{36 - x^2}}$. difficulty ·

2. Find $\displaystyle\int \sqrt{1 - x^2}\, dx$. difficulty ··

3. Find $\displaystyle\int \frac{x^3}{\sqrt{4 - x^2}}\, dx$. difficulty ··

4. Find $\displaystyle\int \frac{dx}{(1 + x^2)^2}$. difficulty ··

5. Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt{x^2 + 1}}$. difficulty ··

6. Find $\displaystyle\int \frac{dx}{x^2\sqrt{x^2 - 4}}$ for $x > 2$. difficulty ··

7. Evaluate $\displaystyle\int_{-2}^1 \frac{dx}{(x^2 + 4x + 13)^{3/2}}$. difficulty ···

8. Find the area inside the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$, and then inside $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. difficulty ···

9. Evaluate $\displaystyle\int_2^4 \frac{\sqrt{x^2 - 4}}{x}\, dx$. difficulty ···

10. The Student-t distribution with 3 degrees of freedom has density $c\left(1 + \dfrac{x^2}{3}\right)^{-2}$ on the whole real line. Find c. difficulty ···

Answers

1. $\frac{\pi}{6} \approx 0.5236$
2. $\frac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right) + C$
3. $\frac{1}{3}(4-x^2)^{3/2} - 4\sqrt{4-x^2} + C$
4. $\frac{1}{2}\arctan x + \dfrac{x}{2(1+x^2)} + C$
5. $\ln(1+\sqrt{2}) \approx 0.8814$
6. $\dfrac{\sqrt{x^2-4}}{4x} + C$
7. $\dfrac{\sqrt{2}}{18} \approx 0.07857$
8. $6\pi \approx 18.85$; in general $\pi a b$
9. $2\sqrt{3} - \dfrac{2\pi}{3} \approx 1.370$
10. $c = \dfrac{2}{\sqrt{3}\,\pi} \approx 0.3676$

Worked solutions

Problem 1

> Evaluate $\displaystyle\int_0^3 \frac{dx}{\sqrt{36 - x^2}}$.

$x = 6\sin\theta$, $dx = 6\cos\theta\, d\theta$, $\sqrt{36 - x^2} = 6\cos\theta$	The form $a^2 - x^2$ with $a = 6$.
$x = 0 \Rightarrow \theta = 0$; $x = 3 \Rightarrow \sin\theta = \frac{1}{2}$, $\theta = \frac{\pi}{6}$	Change the limits with the variable.
$\displaystyle\int_0^{\pi/6} \frac{6\cos\theta}{6\cos\theta}\, d\theta = \int_0^{\pi/6} d\theta$	Everything cancels except $d\theta$.
$\frac{\pi}{6} \approx 0.5236$	The same as $\arcsin\frac{3}{6} - \arcsin 0$, from $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C$.

Problem 2

> Find $\displaystyle\int \sqrt{1 - x^2}\, dx$.

$x = \sin\theta$, $dx = \cos\theta\, d\theta$, $\sqrt{1 - x^2} = \cos\theta$	The form $a^2 - x^2$ with $a = 1$.
$\displaystyle\int \cos\theta \cdot \cos\theta\, d\theta = \int \cos^2\theta\, d\theta$	One $\cos\theta$ from the root and one from dx.
$\displaystyle= \frac{\theta}{2} + \frac{\sin 2\theta}{4} = \frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2}$	$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$, then $\sin 2\theta = 2\sin\theta\cos\theta$ to prepare for the triangle.
$\theta = \arcsin x$, $\sin\theta = x$, $\cos\theta = \sqrt{1 - x^2}$	From the triangle with opposite x and hypotenuse 1.
$\frac{1}{2}\left(\arcsin x + x\sqrt{1 - x^2}\right) + C$	From 0 to 1 this gives $\frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$, the area of a quarter of the unit circle.

Problem 3

Find $\displaystyle\int \frac{x^3}{\sqrt{4-x^2}}\,dx$.

$x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$, $\sqrt{4-x^2} = 2\cos\theta$	The form $a^2 - x^2$ with $a = 2$.
$\displaystyle\int \frac{8\sin^3\theta \cdot 2\cos\theta}{2\cos\theta}\,d\theta = 8\int \sin^3\theta\,d\theta$	$x^3 = 8\sin^3\theta$.
$\displaystyle 8\int (1-\cos^2\theta)\sin\theta\,d\theta = -8\cos\theta + \tfrac{8}{3}\cos^3\theta$	Save one $\sin\theta$ and substitute $w = \cos\theta$, $dw = -\sin\theta\,d\theta$.
$\cos\theta = \frac{\sqrt{4-x^2}}{2}$, so $8\cos\theta = 4\sqrt{4-x^2}$ and $\frac{8}{3}\cos^3\theta = \frac{1}{3}(4-x^2)^{3/2}$	Back to x with the triangle.
$\frac{1}{3}(4-x^2)^{3/2} - 4\sqrt{4-x^2} + C$	Another route: $u = 4 - x^2$, $x^2 = 4 - u$ and $x\,dx = -\frac{1}{2}du$ give the same answer without any trig. An odd power of x in front of the root always allows it.

Problem 4

Find $\displaystyle\int \frac{dx}{(1+x^2)^2}$.

$x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, $1 + x^2 = \sec^2\theta$	The form $a^2 + x^2$ with $a = 1$; there is no root, but the identity still helps.
$\displaystyle\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta = \int \cos^2\theta\,d\theta = \frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2}$	As in Problem 2.
$\displaystyle\sin\theta = \frac{x}{\sqrt{1+x^2}}$, $\cos\theta = \frac{1}{\sqrt{1+x^2}}$, so $\sin\theta\cos\theta = \frac{x}{1+x^2}$	The triangle has opposite x, adjacent 1 and hypotenuse $\sqrt{1+x^2}$.

$\frac{1}{2}\arctan x + \dfrac{x}{2(1+x^2)} + C$	Over the whole line the second term vanishes at both ends, so $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{1}{2}\cdot\pi = \frac{\pi}{2}$.

Problem 5

Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt{x^2+1}}$.

$x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, $\sqrt{x^2+1} = \sec\theta$	The form $a^2 + x^2$ with $a = 1$.
$x = 0 \Rightarrow \theta = 0$; $x = 1 \Rightarrow \theta = \frac{\pi}{4}$	New limits.
$\displaystyle\int_0^{\pi/4} \frac{\sec^2\theta}{\sec\theta}\,d\theta = \int_0^{\pi/4} \sec\theta\,d\theta = \Big[\ln\lvert\sec\theta + \tan\theta\rvert\Big]_0^{\pi/4}$	The standard antiderivative of $\sec\theta$.
$\ln(\sqrt{2}+1) - \ln(1+0)$	$\sec\frac{\pi}{4} = \sqrt{2}$, $\tan\frac{\pi}{4} = 1$.
$\ln(1+\sqrt{2}) \approx 0.8814$	In terms of x the antiderivative is $\ln\left(x + \sqrt{x^2+1}\right)$, which is $\sinh^{-1} x$.

Problem 6

Find $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-4}}$ for $x > 2$.

$x = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-4} = 2\tan\theta$	The form $x^2 - a^2$ with $a = 2$; $0 < \theta < \frac{\pi}{2}$.
$\displaystyle\int \frac{2\sec\theta\tan\theta}{4\sec^2\theta\cdot 2\tan\theta}\,d\theta = \frac{1}{4}\int \cos\theta\,d\theta = \frac{1}{4}\sin\theta$	$\tan\theta$ cancels, and $\frac{1}{\sec\theta} = \cos\theta$.
$\sec\theta = \frac{x}{2}$: hypotenuse x, adjacent 2, opposite $\sqrt{x^2-4}$, so $\sin\theta = \dfrac{\sqrt{x^2-4}}{x}$	sec is hypotenuse over adjacent.

$\dfrac{\sqrt{x^2-4}}{4x}+C$	Check: the derivative is $\frac{1}{4}\cdot\frac{x\cdot\frac{x}{\sqrt{x^2-4}}-\sqrt{x^2-4}}{x^2}=\frac{1}{x^2\sqrt{x^2-4}}$.

Problem 7

Evaluate $\displaystyle\int_{-2}^{1}\frac{dx}{(x^2+4x+13)^{3/2}}$.

$x^2+4x+13=(x+2)^2+9$	Complete the square: $(x+2)^2=x^2+4x+4$, so add 9 to reach 13.
$u=x+2$: $\displaystyle\int_0^3\frac{du}{(u^2+9)^{3/2}}$	A shift; the limits -2 and 1 become 0 and 3.
$u=3\tan\theta$, $du=3\sec^2\theta\,d\theta$, $(u^2+9)^{3/2}=27\sec^3\theta$; θ from 0 to $\frac{\pi}{4}$	The form a^2+u^2 with $a=3$; $u=3$ gives $\tan\theta=1$.
$\displaystyle\int_0^{\pi/4}\frac{3\sec^2\theta}{27\sec^3\theta}\,d\theta=\frac{1}{9}\int_0^{\pi/4}\cos\theta\,d\theta=\frac{1}{9}\sin\frac{\pi}{4}$	$\frac{3}{27}=\frac{1}{9}$.
$\dfrac{\sqrt{2}}{18}\approx 0.07857$	$\frac{1}{9}\cdot\frac{\sqrt{2}}{2}$.

Problem 8

Find the area inside the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$, and then inside $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

The top half is $y=\frac{2}{3}\sqrt{9-x^2}$, so the area is $4\displaystyle\int_0^3\frac{2}{3}\sqrt{9-x^2}\,dx$	Four symmetric quarters.
$x=3\sin\theta$, θ from 0 to $\frac{\pi}{2}$: $\displaystyle\int_0^3\sqrt{9-x^2}\,dx=\int_0^{\pi/2}9\cos^2\theta\,d\theta=\frac{9\pi}{4}$	$\int_0^{\pi/2}\cos^2\theta\,d\theta=\frac{\pi}{4}$: the $\sin 2\theta$ term is 0 at both ends.
$4\cdot\frac{2}{3}\cdot\frac{9\pi}{4}=6\pi$	The area for $a=3$, $b=2$.

In general $4 \cdot \dfrac{b}{a}\displaystyle\int_0^a \sqrt{a^2 - x^2}\,dx = 4 \cdot \dfrac{b}{a} \cdot \dfrac{\pi a^2}{4}$	The same substitution with $x = a\sin\theta$.
$6\pi \approx 18.85$; **in general** πab	With $a = b = r$ it is the circle's πr^2.

Problem 9

> Evaluate $\displaystyle\int_2^4 \frac{\sqrt{x^2 - 4}}{x}\,dx$.

$x = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2 - 4} = 2\tan\theta$	The form $x^2 - a^2$ with $a = 2$.
$x = 2 \Rightarrow \theta = 0$; $x = 4 \Rightarrow \sec\theta = 2$, $\theta = \frac{\pi}{3}$	$\cos\theta = \frac{1}{2}$.
$\displaystyle\int_0^{\pi/3} \frac{2\tan\theta}{2\sec\theta} \cdot 2\sec\theta\tan\theta\,d\theta = 2\int_0^{\pi/3} \tan^2\theta\,d\theta$	$\sec\theta$ cancels.
$\displaystyle 2\int_0^{\pi/3} (\sec^2\theta - 1)\,d\theta = 2\Big[\tan\theta - \theta\Big]_0^{\pi/3}$	Rewrite $\tan^2\theta$ with the identity; $\sec^2\theta$ has antiderivative $\tan\theta$.
$2\sqrt{3} - \dfrac{2\pi}{3} \approx 1.370$	$\tan\frac{\pi}{3} = \sqrt{3}$.

Problem 10

> The Student-t distribution with 3 degrees of freedom has density $c\left(1 + \dfrac{x^2}{3}\right)^{-2}$ on the whole real line. Find c.

c is 1 over $\displaystyle\int_{-\infty}^{\infty} \left(1 + \frac{x^2}{3}\right)^{-2} dx$	A density has to integrate to 1.
$x = \sqrt{3}\tan\theta$, $dx = \sqrt{3}\sec^2\theta\,d\theta$, $1 + \frac{x^2}{3} = \sec^2\theta$; θ runs over $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	As $x \to \pm\infty$, $\theta \to \pm\frac{\pi}{2}$, so the improper integral becomes an ordinary one.

$\int_{-\pi/2}^{\pi/2} \frac{\sqrt{3}\sec^2\theta}{\sec^4\theta}\,d\theta = \sqrt{3}\int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta = \sqrt{3}\cdot\frac{\pi}{2}$	The same $\int \cos^2\theta$ as Problem 4.
$c = \dfrac{2}{\sqrt{3}\,\pi} \approx 0.3676$	This matches the general formula $\frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\,\Gamma(\nu/2)}$ at $\nu = 3$. The tails fall off like x^{-4}, much more slowly than a Gaussian's, which is why a t likelihood is robust to outliers.

Where this goes wrong

1. Leaving out the dx

After substituting for x in the integrand, dx is swapped for $d\theta$ as if the two were the same.

$\int_0^1 \sqrt{1-x^2}\,dx$ with $x = \sin\theta$, so $\sqrt{1-x^2} = \cos\theta$ and θ runs from 0 to $\frac{\pi}{2}$	Right so far: Problem 2, on $[0, 1]$.
"Replace dx with $d\theta$."	The habit that causes the mistake: the substitution applied to the integrand only.
$\int_0^{\pi/2} \cos\theta\,d\theta = 1$ ✗	$dx = \cos\theta\,d\theta$, so the integrand is $\cos^2\theta$, and the area of a quarter of the unit circle is $\frac{\pi}{4} \approx 0.785$, not 1.

2. Keeping the x-limits

The new integral in θ is evaluated at the numbers that were on the original integral sign.

$\int_0^3 \frac{dx}{\sqrt{36-x^2}} = \int d\theta$ with $x = 6\sin\theta$	Right so far: Problem 1, step 3.
"The limits are 0 and 3."	The habit that causes the mistake: the limits copied across without changing variable.

$\int_0^3 d\theta = 3$ ✗	3 is a value of x. At $x = 3$, $\sin\theta = \frac{1}{2}$ and $\theta = \frac{\pi}{6}$, so the answer is $\frac{\pi}{6} \approx 0.524$. Either change the limits or return to x, as $\arcsin\frac{x}{6}$, before evaluating.

3. Completing the square without balancing

The square is built from the x^2 and x terms, and the constant is left as it was.

$x^2 + 4x + 13$ contains $x^2 + 4x$, which is the start of $(x + 2)^2$	Right so far: Problem 7, step 1.
"So $x^2 + 4x + 13 = (x + 2)^2 + 13$."	The habit that causes the mistake: $(x + 2)^2$ also contains $+4$, which has to be taken back out.
$\int_0^3 \frac{du}{(u^2 + 13)^{3/2}} = \frac{3}{13\sqrt{22}} \approx 0.0492$ ✗	$(x + 2)^2 + 13 = x^2 + 4x + 17$, a different quadratic. The correct form is $(x + 2)^2 + 9$, and the integral is $\frac{\sqrt{2}}{18} \approx 0.0786$. Expand the completed square to check it.

4. Reading the wrong side off the triangle

With three sides and six trig functions, it is easy to pick the ratio for the wrong one.

$\int \frac{dx}{x^2\sqrt{x^2 - 4}} = \frac{1}{4}\sin\theta + C$ with $x = 2\sec\theta$	Right so far: Problem 6, step 2.
"$\sec\theta = \frac{x}{2}$, so $\sin\theta = \frac{2}{x}$."	The habit that causes the mistake: the substitution's ratio flipped and used for a different function. $\frac{2}{x}$ is $\cos\theta$.
The answer is $\frac{1}{4} \cdot \frac{2}{x} = \frac{1}{2x} + C$ ✗	Differentiate it: $-\frac{1}{2x^2}$ is negative, but the integrand is positive. At $x = 4$ the derivative is -0.03125 and the integrand is $\frac{1}{16\sqrt{12}} \approx 0.0180$. With hypotenuse x and adjacent 2, the opposite side is $\sqrt{x^2 - 4}$ and $\sin\theta = \frac{\sqrt{x^2-4}}{x}$.

5. Integrating tan²θ by the power rule

$\tan^2\theta$ looks like u^2, and $\int u^2 = \frac{u^3}{3}$.

$\displaystyle\int_2^4 \frac{\sqrt{x^2-4}}{x}\,dx = 2\int_0^{\pi/3} \tan^2\theta\,d\theta$	Right so far: Problem 9, step 3.
"$\int \tan^2\theta\,d\theta = \frac{1}{3}\tan^3\theta$."	The habit that causes the mistake: the power rule needs the derivative of $\tan\theta$, which is $\sec^2\theta$, alongside it, and there is none here.
$2\cdot\frac{1}{3}\tan^3\frac{\pi}{3} = \frac{2}{3}(\sqrt{3})^3 = 2\sqrt{3} \approx 3.464$ ✗	Differentiating $\frac{1}{3}\tan^3\theta$ gives $\tan^2\theta\sec^2\theta$, not $\tan^2\theta$. Use $\tan^2\theta = \sec^2\theta - 1$: the answer is $2\sqrt{3} - \frac{2\pi}{3} \approx 1.370$.

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