Trig identities

Ten trigonometric identities proved from the Pythagorean identity and the sum formulas: double and triple angles, power reduction, half-angle tangent, sum-to-product, tan of a sum, and writing a sin x + b cos x as a single sine, with worked solutions and the mistakes of working both sides and linearizing sin(x + y).

BEFORE YOU START

Two facts, the Pythagorean identity and the sum formulas, are enough to derive every identity on this page. The ten problems build the double-angle, power-reduction, half-angle and sum-to-product formulas from them, then use those to write $3\sin x + 4\cos x$ as a single sine wave.

- x and y are angles, in radians. $\sin^2 x$ means $(\sin x)^2$; $\tan x = \sin x / \cos x$ and $\sec x = 1/\cos x$, both defined where $\cos x \neq 0$.
- Pythagorean identity: $\sin^2 x + \cos^2 x = 1$ for every x.
- Sum formulas: $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$ and $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$, for every x and y; take the upper signs together or the lower signs together.
- Everything else on this page is derived from these. An identity with a denominator or a tan holds only where every term is defined, and each one states that domain.
- To prove an identity, transform one side until it becomes the other; adding to or multiplying both sides *of the identity being proved* assumes what you are proving — that is mistake 1 below.

Problems

1. Prove $\tan^2 x + 1 = \sec^2 x$ from $\sin^2 x + \cos^2 x = 1$. difficulty ·

2. Derive $\sin 2x = 2\sin x \cos x$ from the sum formula, and the three forms of $\cos 2x$. difficulty ·

3. Prove $\cos^2 x = \dfrac{1 + \cos 2x}{2}$ and $\sin^2 x = \dfrac{1 - \cos 2x}{2}$. difficulty ··

4. Prove $\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y$. difficulty ··

5. Prove $\dfrac{1-\cos x}{\sin x} = \tan\dfrac{x}{2}$ for $\sin x \neq 0$. difficulty ··

6. Prove $\sin x + \sin y = 2\sin\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$. difficulty ··

7. Prove $\dfrac{\sin x}{1+\cos x} + \dfrac{1+\cos x}{\sin x} = \dfrac{2}{\sin x}$, and say where it is valid. difficulty ···

8. Derive $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$ from the sum formulas. difficulty ···

9. Prove $\cos 3x = 4\cos^3 x - 3\cos x$. difficulty ···

10. Write $3\sin x + 4\cos x$ as $R\sin(x+\varphi)$ with $R > 0$, giving R and φ exactly. difficulty ···

Answers

1. $\tan^2 x + 1 = \sec^2 x$
2. $\sin 2x = 2\sin x \cos x$; $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
3. $\cos^2 x = \frac{1+\cos 2x}{2}$, $\sin^2 x = \frac{1-\cos 2x}{2}$
4. $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y$
5. $\tan \frac{x}{2}$
6. $2\sin \frac{x+y}{2} \cos \frac{x-y}{2}$
7. $\dfrac{2}{\sin x}$, valid when $\sin x \neq 0$ (which also makes $1 + \cos x \neq 0$)
8. $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$, for $\cos x, \cos y \neq 0$ and $\tan x \tan y \neq 1$
9. $\cos 3x = 4\cos^3 x - 3\cos x$
10. $R = 5$, $\varphi = \arctan \frac{4}{3}$ (first quadrant, since $\cos\varphi = \frac{3}{5} > 0$ and $\sin\varphi = \frac{4}{5} > 0$)

Worked solutions

Problem 1

> Prove $\tan^2 x + 1 = \sec^2 x$ from $\sin^2 x + \cos^2 x = 1$.

Take any x with $\cos x \neq 0$.	$\tan x$ and $\sec x$ are defined only where $\cos x \neq 0$, so the identity is a claim about those angles alone.
$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$	Both terms of the Pythagorean identity have a factor $\cos^2 x$ to divide by, which is nonzero where $\tan x$ is defined, and dividing a true equation by a nonzero number gives a true equation. This starts from a known fact, not from the identity to be proved.
$\tan^2 x + 1 = \sec^2 x$	$\sin^2 x / \cos^2 x = (\sin x / \cos x)^2 = \tan^2 x$, $\cos^2 x / \cos^2 x = 1$ and $1 / \cos^2 x = \sec^2 x$; valid wherever $\cos x \neq 0$.

Problem 2

> Derive $\sin 2x = 2 \sin x \cos x$ from the sum formula, and the three forms of $\cos 2x$.

$\sin 2x = \sin(x + x) = \sin x \cos x + \cos x \sin x.$	The sine sum formula holds for every pair of angles, so it holds with $y = x$, and $x + x = 2x$.
$\sin 2x = 2 \sin x \cos x.$	The two terms are the same product written in a different order.
$\cos 2x = \cos(x + x) = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x.$	The cosine sum formula with $y = x$.
$\cos^2 x - \sin^2 x = \cos^2 x - (1 - \cos^2 x) = 2 \cos^2 x - 1.$	The Pythagorean identity gives $\sin^2 x = 1 - \cos^2 x$; substituting it leaves a form with cosines only.

$\cos^2 x - \sin^2 x = (1 - \sin^2 x) - \sin^2 x = 1 - 2\sin^2 x.$	Likewise $\cos^2 x = 1 - \sin^2 x$ leaves a form with sines only.
$\sin 2x = 2\sin x \cos x$; $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$	Steps 2 to 5. Which form of $\cos 2x$ to use depends on which function the target keeps: Problems 3, 5 and 9 pick the form that suits their target.

Problem 3

> Prove $\cos^2 x = \dfrac{1 + \cos 2x}{2}$ and $\sin^2 x = \dfrac{1 - \cos 2x}{2}$.

$\cos 2x = 2\cos^2 x - 1.$	Problem 2's second form is the one whose only square is $\cos^2 x$, so solving it for $\cos^2 x$ gives the first identity. It is already proved, so rearranging it is a deduction from a true equation, not an assumption.
$2\cos^2 x = 1 + \cos 2x$, so $\cos^2 x = \frac{1+\cos 2x}{2}$.	Add 1 to both sides, then divide both by 2; each step keeps a true equation true.
$\cos 2x = 1 - 2\sin^2 x.$	Problem 2's third form is the one whose only square is $\sin^2 x$.
$2\sin^2 x = 1 - \cos 2x$, so $\sin^2 x = \frac{1-\cos 2x}{2}$.	Add $2\sin^2 x - \cos 2x$ to both sides, then divide both by 2.
$\cos^2 x = \frac{1+\cos 2x}{2}$, $\sin^2 x = \frac{1-\cos 2x}{2}$	Steps 2 and 4. Each trades a square for a first power of a cosine at twice the angle, which is how $\int \cos^2 x\,dx$ and $\int \sin^2 x\,dx$ are done.

Problem 4

> Prove $\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y$.

$\sin(x + y) = \sin x \cos y + \cos x \sin y$ and $\sin(x - y) = \sin x \cos y - \cos x \sin y.$	The sine sum formula with the upper and with the lower sign.

$\sin(x+y)\sin(x-y) = \sin^2 x \cos^2 y - \cos^2 x \sin^2 y.$	The two factors are the sum and the difference of the same two products, $\sin x \cos y$ and $\cos x \sin y$, and a sum times a difference is the difference of the squares.
$= \sin^2 x\,(1 - \sin^2 y) - (1 - \sin^2 x)\sin^2 y.$	The target contains only sines, so replace each $\cos^2$ by $1 - \sin^2$ (Pythagorean identity).
$= \sin^2 x - \sin^2 x \sin^2 y - \sin^2 y + \sin^2 x \sin^2 y.$	Multiply out both products.
$\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y$	The two $\sin^2 x \sin^2 y$ terms cancel. The left side has been transformed into the right side.

Problem 5

> Prove $\dfrac{1-\cos x}{\sin x} = \tan\dfrac{x}{2}$ for $\sin x \neq 0$.

$x = 2 \cdot \frac{x}{2}.$	The right side is a function of the angle $\frac{x}{2}$, so write x as twice that angle and use Problem 2, whose identities hold for every angle, at $\frac{x}{2}$.
$1 - \cos x = 1 - \left(1 - 2\sin^2 \frac{x}{2}\right) = 2\sin^2 \frac{x}{2}.$	Problem 2's form $\cos 2x = 1 - 2\sin^2 x$, at the angle $\frac{x}{2}$, gives $\cos x = 1 - 2\sin^2 \frac{x}{2}$; this form is chosen because its 1 cancels the 1 in the numerator.
$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}.$	Problem 2's $\sin 2x = 2 \sin x \cos x$ at the angle $\frac{x}{2}$.
$\dfrac{1-\cos x}{\sin x} = \dfrac{2\sin^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}}.$	Steps 2 and 3 in the numerator and denominator. The denominator is $\sin x$, which is nonzero by hypothesis, so neither $\sin \frac{x}{2}$ nor $\cos \frac{x}{2}$ is 0.

$\tan \frac{x}{2}$	Cancel $2 \sin \frac{x}{2}$, nonzero by step 4, leaving $\sin \frac{x}{2} / \cos \frac{x}{2}$. Since $\cos \frac{x}{2} \neq 0$, $\tan \frac{x}{2}$ is defined, so both sides exist for every x with $\sin x \neq 0$.

Problem 6

Prove $\sin x + \sin y = 2 \sin \dfrac{x+y}{2} \cos \dfrac{x-y}{2}$.

Let $u = \frac{x+y}{2}$ and $v = \frac{x-y}{2}$, the half-sum and half-difference of x and y.	The right side is written in these two angles, so naming them lets the left side be expanded with the sum formulas.
$u + v = x$ and $u - v = y$.	Adding the two definitions gives $\frac{2x}{2} = x$; subtracting them gives $\frac{2y}{2} = y$. So every pair x, y is a sum and a difference of u and v.
$\sin x = \sin(u+v) = \sin u \cos v + \cos u \sin v$.	Step 2, then the sine sum formula with the upper sign.
$\sin y = \sin(u-v) = \sin u \cos v - \cos u \sin v$.	Step 2, then the sine sum formula with the lower sign.
$\sin x + \sin y = 2 \sin u \cos v$.	Add steps 3 and 4: the $\cos u \sin v$ terms cancel and the $\sin u \cos v$ terms double.
$2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$	Put back $u = \frac{x+y}{2}$ and $v = \frac{x-y}{2}$ from step 1.

Problem 7

Prove $\dfrac{\sin x}{1+\cos x} + \dfrac{1+\cos x}{\sin x} = \dfrac{2}{\sin x}$, and say where it is valid.

The identity needs $\sin x \neq 0$ and $1 + \cos x \neq 0$; the first implies the second.	Both are denominators. If $\cos x = -1$ then $\sin^2 x = 1 - \cos^2 x = 0$ by the Pythagorean identity, so excluding $\sin x = 0$ also excludes $\cos x = -1$.

$\dfrac{\sin x}{1+\cos x} + \dfrac{1+\cos x}{\sin x} = \dfrac{\sin^2 x + (1+\cos x)^2}{(1+\cos x)\sin x}.$	Common denominator: multiply the top and bottom of each fraction by the other's denominator, which is nonzero by step 1.
$\sin^2 x + (1+\cos x)^2 = \sin^2 x + 1 + 2\cos x + \cos^2 x = 2 + 2\cos x.$	Expand the square; $\sin^2 x + \cos^2 x = 1$ collapses two of the terms to 1.
$\dfrac{2+2\cos x}{(1+\cos x)\sin x} = \dfrac{2(1+\cos x)}{(1+\cos x)\sin x}.$	Factoring the 2 out of the numerator exposes the factor $1+\cos x$ it shares with the denominator.
$\dfrac{2}{\sin x}$, **valid when** $\sin x \neq 0$ **(which also makes** $1+\cos x \neq 0$**)**	Cancel $1+\cos x$, nonzero by step 1. So the identity holds for every x except $x = k\pi$, k an integer, where $\sin x = 0$.

Problem 8

Derive $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$ from the sum formulas.

$\tan(x+y) = \dfrac{\sin(x+y)}{\cos(x+y)} = \dfrac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}.$	The definition of tan, valid where $\cos(x+y) \neq 0$, then the two sum formulas with the upper sign.
Suppose $\cos x \neq 0$ and $\cos y \neq 0$.	These are the conditions for $\tan x$ and $\tan y$ on the right side to be defined, and they make $\cos x \cos y$ a nonzero number to divide by.
$\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}.$	Dividing the top and bottom of a fraction by the same nonzero number, $\cos x \cos y$, leaves it unchanged: $\frac{\sin x \cos y}{\cos x \cos y} = \tan x$, $\frac{\cos x \sin y}{\cos x \cos y} = \tan y$, $\frac{\cos x \cos y}{\cos x \cos y} = 1$ and $\frac{\sin x \sin y}{\cos x \cos y} = \tan x \tan y$.
$1 - \tan x \tan y = \dfrac{\cos(x+y)}{\cos x \cos y}.$	Step 3 divided the denominator of step 1, which is $\cos(x+y)$, by $\cos x \cos y$. So $\tan x \tan y \neq 1$ is the same condition as $\cos(x+y) \neq 0$, the one step 1 needed.

$\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$, **for** $\cos x, \cos y \neq 0$ **and** $\tan x \tan y \neq 1$	Steps 1 to 4: the first two conditions make the right side's tangents exist, and the third makes both sides' denominators nonzero.

Problem 9

> Prove $\cos 3x = 4\cos^3 x - 3\cos x$.

$\cos 3x = \cos(2x + x) = \cos 2x \cos x - \sin 2x \sin x.$	$3x = 2x + x$, and the cosine sum formula with the angles $2x$ and x produces $\cos 2x$ and $\sin 2x$, which Problem 2 already expresses in $\sin x$ and $\cos x$.
$= (2\cos^2 x - 1)\cos x - (2\sin x \cos x)\sin x.$	Problem 2. Of the three forms of $\cos 2x$, take $2\cos^2 x - 1$, because the target contains only cosines.
$= 2\cos^3 x - \cos x - 2\sin^2 x \cos x.$	Multiply out.
$= 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x = 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x.$	The Pythagorean identity $\sin^2 x = 1 - \cos^2 x$ removes the last sine; then multiply out.
$\cos 3x = 4\cos^3 x - 3\cos x$	Collect the $\cos^3 x$ terms and the $\cos x$ terms. A check at $x = 0$: $\cos 0 = 1$ and $4 - 3 = 1$.

Problem 10

> Write $3\sin x + 4\cos x$ as $R\sin(x + \varphi)$ with $R > 0$, giving R and φ exactly.

$R\sin(x+\varphi) = R\sin x\cos\varphi + R\cos x\sin\varphi = (R\cos\varphi)\sin x + (R\sin\varphi)\cos x.$	The sine sum formula with the angles x and φ; R and φ are constants, so $R\cos\varphi$ and $R\sin\varphi$ are fixed coefficients.

$R\cos\varphi = 3$ and $R\sin\varphi = 4$.	Matching the coefficients of $\sin x$ and $\cos x$ with $3\sin x + 4\cos x$ makes the two expressions equal for every x, and the match is forced: $x = 0$ gives $R\sin\varphi = 4$ and $x = \frac{\pi}{2}$ gives $R\cos\varphi = 3$.
$R^2 = R^2\cos^2\varphi + R^2\sin^2\varphi = 3^2 + 4^2 = 25$, so $R = 5$.	Square the two equations of step 2 and add; the Pythagorean identity removes φ. $R > 0$ picks the positive root.
$\cos\varphi = \frac{3}{5}$, $\sin\varphi = \frac{4}{5}$, so $\tan\varphi = \frac{4}{3}$.	Divide both equations of step 2 by $R = 5$, then divide the second by the first.
$R = 5$, $\varphi = \arctan\frac{4}{3}$ **(first quadrant, since** $\cos\varphi = \frac{3}{5} > 0$ **and** $\sin\varphi = \frac{4}{5} > 0$**)**	$\tan\varphi = \frac{4}{3}$ alone has two solutions in a full turn: $\arctan\frac{4}{3}$ in the first quadrant and $\arctan\frac{4}{3} + \pi$ in the third, where sine and cosine are both negative. Both are positive here, so φ is the first-quadrant one, which is the one $\arctan$ returns (about 0.927 rad; adding any multiple of 2π gives the same sine). So $3\sin x + 4\cos x = 5\sin\left(x + \arctan\frac{4}{3}\right)$, a sine wave of amplitude 5.

Where this goes wrong

1. Working on both sides

Solving an equation means doing the same thing to both sides, and it is natural to treat an identity to be proved the same way.

To prove: $\dfrac{1-\cos x}{\sin x} = \tan\dfrac{x}{2}$ for $\sin x \neq 0$	Right so far: the statement of Problem 5.
“Multiply both sides by the denominators and simplify until something obviously true appears.”	The analogy that causes the mistake: solving an equation, where operating on both sides is exactly right.

$(1-\cos x)\cos\frac{x}{2} = \sin x \sin\frac{x}{2}$, so $2\sin^2\frac{x}{2}\cos\frac{x}{2} = 2\sin^2\frac{x}{2}\cos\frac{x}{2}$, so $0 = 0$ ✗	The first equation is the identity multiplied by $\sin x \cos\frac{x}{2}$, so every line after it takes the identity as true to begin with. Reaching a true statement from a claim does not make the claim true: $1 = 2$ multiplied by 0 also gives $0 = 0$. The chain proves the identity only if every step reverses, so that it can be run backward from $0 = 0$ to the claim. In this particular case the steps do reverse — the multiplier $\sin x \cos\frac{x}{2}$ is nonzero whenever $\sin x \neq 0$, so dividing by it undoes the first step — and the identity is true (Problem 5). The *method* is the error, not the conclusion: written this way, nothing was proved, and the same method "proves" false identities. Transform one side into the other instead.

2. Linearizing the sine of a sum

Many operations split over a sum: $2(x+y) = 2x + 2y$, and any linear map does the same.

$\sin(x+y)$ for two angles x and y	Right so far: the expression to expand.
"The sine of a sum is the sum of the sines."	The analogy that causes the mistake: treating $\sin$ as if it were linear.
$\sin(x+y) = \sin x + \sin y$ ✗	$\sin$ is not linear. A check with $x = y = \frac{\pi}{2}$: the left side is $\sin\pi = 0$, the right side is $1 + 1 = 2$. The sum formula has cross terms, $\sin(x+y) = \sin x\cos y + \cos x \sin y$, and Problem 6 shows what $\sin x + \sin y$ actually equals: $2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$.

3. Doubling the angle by doubling the cosine

The same habit, with a factor instead of a sum: doubling the input is expected to double the output.

$\cos 2x$	Right so far: the cosine of twice the angle.
“Twice the angle, so twice the cosine.”	The analogy that causes the mistake: $\cos(2x)$ read as if cos were multiplication by a constant.
$\cos 2x = 2\cos x$ ✗	The same linearity habit. A check with $x = 0$: the left side is $\cos 0 = 1$, the right side is 2. It fails wherever $\cos x > \frac{1}{2}$ without any calculation, since there $2\cos x > 1$ and no cosine exceeds 1. The double-angle forms are $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$ (Problem 2).

4. Dividing by something that can be zero

Once the algebra of a proof works, the conditions it needs are easy to leave off.

$\dfrac{\sin x}{1+\cos x} + \dfrac{1+\cos x}{\sin x}$	Right so far: the left side of Problem 7.
“Common denominator, cancel, done; the domain is a technicality.”	The attitude that causes the mistake: treating the division as always allowed.
$\dfrac{\sin x}{1+\cos x} + \dfrac{1+\cos x}{\sin x} = \dfrac{2(1+\cos x)}{(1+\cos x)\sin x} = \dfrac{2}{\sin x}$ for every x ✗	Every fraction after the first divides by $\sin x$, and $\sin x = 0$ at $x = k\pi$ for every integer k; at odd multiples of π, $1 + \cos x = 0$ as well, so the first fraction is undefined too and the cancellation divides 0 by 0. At those angles neither side is defined, so the identity is false there as a claim “for every x”. The correct statement carries its domain: valid when $\sin x \neq 0$ (Problem 7). Every identity with a denominator does.

5. Losing the sign under a square root

Problem 3 gives $\sin^2 x$, and taking a square root looks like the way to get $\sin x$ itself.

$\sin^2 x = \frac{1-\cos 2x}{2}$	Right so far: Problem 3.
"Take the square root of both sides; a square root undoes a square."	The analogy that causes the mistake: $\sqrt{3^2} = 3$, remembered without its condition that the number squared is not negative.
$\sqrt{\sin^2 x} = \sin x$, so $\sin x = \sqrt{\frac{1-\cos 2x}{2}}$ ✗	A square root is never negative, so $\sqrt{\sin^2 x} = \lvert\sin x\rvert$. At $x = -\frac{\pi}{2}$ the line gives $\sin x = \sqrt{\frac{1-(-1)}{2}} = 1$, but $\sin(-\frac{\pi}{2}) = -1$. The correct result is $\lvert\sin x\rvert = \sqrt{\frac{1-\cos 2x}{2}}$, with the sign read off the quadrant of x. The power-reduction identities are safe precisely because they never take the root.

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