Series convergence tests: geometric, divergence, integral, comparison and alternating

Ten problems on infinite series: a geometric series that starts at n = 1, a telescoping sum, the divergence test and what it cannot say about $1/\sqrt{n}$, the integral test on $1/(n \ln^2 n)$ with bounds on the sum, a direct comparison for $(2 + \cos n)/n^2$, a limit comparison, the alternating harmonic series and conditional convergence, how many terms an alternating error bound needs, the ratio test on $n!/n^n$, and the step-size conditions behind stochastic gradient descent, with worked solutions and the mistakes of starting a geometric sum at the wrong term, reading terms that go to 0 as convergence and comparing with a bigger divergent series.

BEFORE YOU START

An infinite series adds up infinitely many numbers, and the first question is whether the total is a finite number at all. Taylor series, the geometric sums behind an exponential moving average, and the step-size conditions in proofs that stochastic gradient descent converges all depend on the answer. A handful of tests settle most series: compare with a geometric series or a p-series, compare with an integral, or use the alternating signs. The work goes wrong in a few places: a geometric sum started at the wrong term, terms that go to 0 read as proof of convergence, a comparison made in the direction that proves nothing, the alternating series test used on terms that do not shrink, and a ratio limit of 1 treated as a verdict. These ten problems cover each of them.

- **Convergence.** $\sum_{n=1}^{\infty} a_n$ converges to S if the partial sums $S_N = a_1 + \cdots + a_N$ tend to S. Otherwise it diverges.
- **Geometric series.** $\sum_{n=0}^{\infty} ar^n = \dfrac{a}{1-r}$ for $|r| < 1$, and it diverges for $|r| \geq 1$. In general the sum is $\dfrac{\text{first term}}{1-r}$.
- **Divergence test.** If $a_n \not\to 0$, the series diverges. If $a_n \to 0$, this test says nothing.
- **Integral test and p-series.** If f is positive and decreasing with $f(n) = a_n$, then $\sum a_n$ and $\int_1^{\infty} f(x)\,dx$ converge or diverge together, and $\displaystyle\int_N^{\infty} f \leq \sum_{n=N}^{\infty} a_n \leq a_N + \int_N^{\infty} f$. So $\displaystyle\sum \frac{1}{n^p}$ converges exactly when $p > 1$.
- **Comparison.** For $0 \leq a_n \leq b_n$: if $\sum b_n$ converges so does $\sum a_n$; if $\sum a_n$ diverges so does $\sum b_n$. **Limit comparison:** if $a_n, b_n > 0$ and $a_n/b_n \to L$ with $0 < L < \infty$, both converge or both diverge.

- **Alternating series.** If b_n decreases to 0, $\sum (-1)^{n+1} b_n$ converges, and stopping after N terms leaves an error of at most b_{N+1}. A series is **absolutely** convergent if $\sum |a_n|$ converges, and **conditionally** convergent if it converges but $\sum |a_n|$ does not.
- **Ratio test.** If $|a_{n+1}/a_n| \to \ell$, the series converges absolutely for $\ell < 1$ and diverges for $\ell > 1$. At $\ell = 1$ it says nothing.

Builds on: Improper integrals: infinite limits, infinite integrands and comparison, Evaluating limits: factoring, conjugates and limits at infinity

Problems

1. Find $\displaystyle\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^{n}$. difficulty ·

2. Find $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ from its partial sums. difficulty ·

3. What does the divergence test say about $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+1}$, and about $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$? difficulty ·

4. Use the integral test to show that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges, and bound its sum. difficulty ··

5. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{2+\cos n}{n^2}$ converges, and give bounds for its sum. difficulty ··

6. Does $\displaystyle\sum_{n=1}^{\infty} \frac{3n+1}{n^3+2}$ converge? difficulty ··

7. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \dots$ converges, decide whether it converges absolutely, and bound the error of S_4. difficulty ··

8. How many terms of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ guarantee an error less than 0.001? difficulty ··

9. Use the ratio test on $\sum_{n=1}^{\infty} \frac{n!}{n^n}$. difficulty ···

10. Proofs that stochastic gradient descent converges usually ask for step sizes $\alpha_n > 0$ with $\sum \alpha_n = \infty$ and $\sum \alpha_n^2 < \infty$. For which p does $\alpha_n = \frac{c}{n^p}$ (with $c > 0$) qualify? Does $\alpha_n = 0.1 \cdot 0.99^n$? difficulty ···

Answers

1. 10
2. 1
3. The first diverges; for the second the test is silent, and it diverges anyway
4. It converges, with $1.443 \le S \le 2.483$
5. It converges, to a sum between 1.645 and 4.935
6. Yes, it converges
7. It converges conditionally, not absolutely; $|S - S_4| \le 0.2$
8. 10 terms
9. $\ell = \frac{1}{e} \approx 0.368 < 1$, so the series converges
10. $\frac{1}{2} < p \le 1$; the geometric schedule fails, because $\sum \alpha_n = 9.9$ is finite

Worked solutions

Problem 1

> Find $\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^{n}$.

$r = \frac{2}{3}$, and $\lvert r\rvert < 1$, so the series converges	A geometric series: each term is $\frac{2}{3}$ of the one before.
The first term, at $n = 1$, is $5 \cdot \frac{2}{3} = \frac{10}{3}$	The sum starts at $n = 1$, not $n = 0$.
$\dfrac{10/3}{1 - 2/3} = \dfrac{10/3}{1/3}$	First term over $1 - r$.
10	The partial sums $\frac{10}{3}$, $\frac{50}{9}$, $\frac{190}{27}$, … climb toward 10.

Problem 2

> Find $\sum_{n=1}^{\infty} \dfrac{1}{n(n+1)}$ from its partial sums.

$\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$	Partial fractions.
$S_N = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) = 1 - \dfrac{1}{N+1}$	Every middle term cancels with the next one: the sum telescopes.
$1 - \dfrac{1}{N+1} \to 1$	Take the limit of the partial sums.
1	For example $S_{99} = 0.99$. Telescoping sums are one of the few kinds, with geometric series, whose exact value is easy to find.

Problem 3

> What does the divergence test say about $\sum_{n=1}^{\infty} \frac{n}{2n+1}$, and about $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$?

$\frac{n}{2n+1} \to \frac{1}{2} \neq 0$	Divide top and bottom by n.
So $\sum \frac{n}{2n+1}$ diverges	Adding terms near $\frac{1}{2}$ forever gives partial sums that grow like $\frac{N}{2}$.
$\frac{1}{\sqrt{n}} \to 0$, so the divergence test says nothing about $\sum \frac{1}{\sqrt{n}}$	Terms going to 0 are needed for convergence, but they are not enough.
The first diverges; for the second the test is silent, and it diverges anyway	It is a p-series with $p = \frac{1}{2} \leq 1$. Directly: each of the first N terms is at least $\frac{1}{\sqrt{N}}$, so $S_N \geq \sqrt{N}$. $S_{100} \approx 18.59$.

Problem 4

> Use the integral test to show that $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges, and bound its sum.

$f(x) = \frac{1}{x(\ln x)^2}$ is positive and decreasing for $x \geq 2$	Both x and $\ln x$ increase, so the denominator does.
$\int_2^b \frac{dx}{x(\ln x)^2} = \Big[-\frac{1}{\ln x} \Big]_2^b = \frac{1}{\ln 2} - \frac{1}{\ln b}$	Substitute $u = \ln x$, $du = \frac{dx}{x}$: the integral of u^{-2}.
$\frac{1}{\ln b} \to 0$, so the integral is $\frac{1}{\ln 2} \approx 1.4427$ and the series converges	The integral test.
$\frac{1}{\ln 2} \leq S \leq \frac{1}{2(\ln 2)^2} + \frac{1}{\ln 2}$	The bounds with $N = 2$: the integral alone, and the integral plus the first term.

It converges, with $1.443 \le S \le 2.483$	The sum is about 2.110. Without the square, $\sum \frac{1}{n \ln n}$ diverges: its integral is $\ln(\ln b) \to \infty$.

Problem 5

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{2 + \cos n}{n^2}$ converges, and give bounds for its sum.

$1 \le 2 + \cos n \le 3$, so $\dfrac{1}{n^2} \le \dfrac{2 + \cos n}{n^2} \le \dfrac{3}{n^2}$	$\cos n$ is between -1 and 1. Every term is positive.
$\displaystyle\sum \frac{3}{n^2}$ converges	A p-series with $p = 2 > 1$, times 3.
So the series converges by direct comparison	Smaller than a convergent series, term by term.
$\displaystyle\sum \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.645$ and $\displaystyle\sum \frac{3}{n^2} = \frac{\pi^2}{2} \approx 4.935$	The known value of $\sum \frac{1}{n^2}$.
It converges, to a sum between 1.645 **and** 4.935	The sum is about 3.614. The integral test can't be used here, because the terms are not decreasing.

Problem 6

Does $\displaystyle\sum_{n=1}^{\infty} \frac{3n + 1}{n^3 + 2}$ converge?

For large n, $\dfrac{3n + 1}{n^3 + 2}$ behaves like $\dfrac{3n}{n^3} = \dfrac{3}{n^2}$	Keep the highest power on top and bottom.
$\dfrac{(3n + 1)/(n^3 + 2)}{1/n^2} = \dfrac{3n^3 + n^2}{n^3 + 2} \to 3$	Limit comparison with $b_n = \frac{1}{n^2}$.
$0 < 3 < \infty$ and $\sum \frac{1}{n^2}$ converges	Both series do the same thing.

Yes, it converges	A direct comparison works too: $3n + 1 \le 4n$ and $n^3 + 2 > n^3$, so the terms are below $\frac{4}{n^2}$. The sum is about 3.260.

Problem 7

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \dots$ converges, decide whether it converges absolutely, and bound the error of S_4.

$b_n = \frac{1}{n}$ decreases and tends to 0	The conditions of the alternating series test.
So the series converges	Each partial sum overshoots the last in the other direction, by less each time.
$\displaystyle\sum \left\lvert \frac{(-1)^{n+1}}{n} \right\rvert = \sum \frac{1}{n}$ diverges	The harmonic series, a p-series with $p = 1$.
$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12} \approx 0.5833$, with error at most $b_5 = \frac{1}{5}$	The first omitted term.
It converges conditionally, not absolutely; $\lvert S - S_4 \rvert \le 0.2$	The sum is $\ln 2 \approx 0.6931$, so the actual error is about 0.110, inside the bound.

Problem 8

How many terms of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ guarantee an error less than 0.001?

$b_n = \frac{1}{n^3}$ decreases to 0, so the error after N terms is at most $\dfrac{1}{(N+1)^3}$	The alternating series error bound.
$\dfrac{1}{(N+1)^3} < 0.001$ needs $(N+1)^3 > 1000$, so $N + 1 > 10$	$10^3 = 1000$ exactly, which is not enough for a strict inequality.

$N = 10$: the bound is $\frac{1}{1331} \approx 0.00075$	$N = 9$ gives a bound of exactly 0.001.
10 **terms**	$S_{10} \approx 0.90112$, and the sum is $\frac{3}{4}\zeta(3) \approx 0.90154$: the actual error is about 0.00043.

Problem 9

> Use the ratio test on $\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$	Write out the ratio of consecutive terms.
$= \frac{(n+1)\,n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n$	$(n+1)!/n! = n+1$, which cancels one factor of $n+1$.
$\left(\frac{n}{n+1}\right)^n = \frac{1}{(1+1/n)^n} \to \frac{1}{e}$	The limit that defines e.
$\ell = \frac{1}{e} \approx 0.368 < 1$, **so the series converges**	The sum is about 1.880. Factorials and powers in the same term are what the ratio test is for: the comparison tests have nothing simple to compare with.

Problem 10

> Proofs that stochastic gradient descent converges usually ask for step sizes $\alpha_n > 0$ with $\sum \alpha_n = \infty$ and $\sum \alpha_n^2 < \infty$. For which p does $\alpha_n = \frac{c}{n^p}$ (with $c > 0$) qualify? Does $\alpha_n = 0.1 \cdot 0.99^n$?

$\sum \frac{c}{n^p} = \infty$ exactly when $p \le 1$	The p-series test.
$\sum \left(\frac{c}{n^p}\right)^2 = c^2 \sum \frac{1}{n^{2p}} < \infty$ exactly when $2p > 1$	Again the p-series test, now with exponent $2p$.

Both hold when $\frac{1}{2} < p \leq 1$	So $\frac{c}{n}$ qualifies, and $\frac{c}{\sqrt{n}}$ (whose squares form the harmonic series) and $\frac{c}{n^2}$ (whose sum is finite) do not.
$\sum_{n=1}^{\infty} 0.1 \cdot 0.99^n = \frac{0.1 \cdot 0.99}{1 - 0.99} = 9.9$	Geometric, with first term 0.099.
$\frac{1}{2} < p \leq 1$; **the geometric schedule fails, because $\sum \alpha_n = 9.9$ is finite**	The first condition lets the steps reach a minimum however far away it starts; with steps adding to 9.9, the iterates can only travel 9.9 times the largest gradient. The second condition makes the noise in the gradients average out.

Where this goes wrong

1. Starting the geometric sum at the wrong term

$\frac{a}{1 - r}$ is memorized with a as the number in front of r^n.

$\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^n$, geometric with $r = \frac{2}{3}$	Right so far: Problem 1, step 1.
"$a = 5$."	The habit that causes the mistake: 5 is the term at $n = 0$, and this sum starts at $n = 1$.
$\frac{5}{1 - 2/3} = 15$ ✗	That includes a term of 5 that is not in the series. The formula needs the first term actually in the sum, $5 \cdot \frac{2}{3} = \frac{10}{3}$, and the sum is 10.

2. Reading terms that go to 0 as convergence

The divergence test gets used backwards.

$\frac{1}{\sqrt{n}} \to 0$	Right so far: Problem 3, step 3.

"The terms go to 0, so the series converges."	The habit that causes the mistake: the divergence test turned around.
$\sum \frac{1}{\sqrt{n}}$ converges ✗	It diverges: $S_N \geq \sqrt{N}$, and $S_{10000} \approx 198.5$. Terms that go to 0 are necessary for convergence, not sufficient, and the harmonic series $\sum \frac{1}{n}$ is the standard example.

3. Comparing with a bigger divergent series

Any true inequality looks like a comparison.

$a_n = \frac{3n+1}{n^3+2} > 0$	Right so far: Problem 6.
"$a_n \leq \frac{4}{n}$ for every $n \geq 1$, and $\sum \frac{4}{n}$ diverges."	The inequality is true: $(3n+1)n \leq 4(n^3+2)$ for every $n \geq 1$.
So $\sum a_n$ diverges ✗	Being smaller than a divergent series proves nothing; a series of zeros is smaller too. To prove divergence, a_n must be **at least** the terms of a divergent series; to prove convergence, at most those of a convergent one. Here $a_n \leq \frac{4}{n^2}$, and the series converges.

4. Using the alternating series test on terms that don't shrink to 0

The signs alternate, so the test seems to apply.

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+1}$, with $b_n = \frac{n}{n+1}$	The setup of Problem 7, on a different series.
"It alternates, so it converges."	The habit that causes the mistake: the test has two conditions, and only the alternation was checked.

The series converges ✗	$b_n \to 1$, not 0, so the terms do not go to 0 and the series diverges by the divergence test: its partial sums keep jumping by nearly 1. The alternating series test needs b_n to decrease **to** 0.

5. Taking a ratio limit of 1 as an answer

The ratio test is applied to every series, and its answer is read even at the boundary.

For $\sum \frac{1}{n}$: $\dfrac{a_{n+1}}{a_n} = \dfrac{n}{n+1} \to 1$	Right so far: the same step as Problem 9, step 1.
"$\ell \leq 1$, so it converges."	The habit that causes the mistake: the boundary case lumped in with $\ell < 1$.
$\sum \frac{1}{n}$ converges ✗	It diverges. At $\ell = 1$ the ratio test gives no answer: $\sum \frac{1}{n^2}$ also has ratio $\left(\frac{n}{n+1}\right)^2 \to 1$ and converges. Any p-series has ratio limit 1, so for these, use the integral test or a comparison instead.

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