Separable differential equations: growth, decay, cooling and the logistic curve

Ten separable differential equations: exponential growth with a negative start, y′ = x/y, a solution that blows up in finite time, the equation whose solution is softplus, half-life, Newton's law of cooling, the sigmoid as the solution of y′ = y(1 − y), gradient flow on a quadratic loss, a constant solution lost by dividing, and a draining tank, with worked solutions and the mistakes of adding the constant too late, splitting a square root and forgetting where a solution stops.

BEFORE YOU START

A differential equation describes how a quantity changes, and solving it recovers the quantity itself. When the rate factors into a function of x times a function of y, the variables can be separated and each side integrated on its own. The sigmoid, softplus and the exponential decay of gradient flow on a quadratic loss all come out of equations like this. The work goes wrong in a few places: the constant of integration added after exponentiating instead of before, $\sqrt{x^2+9}$ split into $x+3$, a constant solution thrown away by dividing by zero, the shift in Newton's law of cooling forgotten, and a formula used past the point where the solution stops. These ten problems cover each of them.

- **Separable.** $\dfrac{dy}{dx} = g(x)\,h(y)$. Divide by $h(y)$, multiply by dx, and integrate both sides: $\displaystyle\int \frac{dy}{h(y)} = \int g(x)\,dx + C$. One constant is enough, and it goes in **before** you solve for y.
- **Constant solutions.** If $h(y_0) = 0$, then $y \equiv y_0$ is a solution. Dividing by $h(y)$ assumes $h(y) \neq 0$, so check these separately.
- **Exponential growth and decay.** $y' = ky$ has solutions $y = y(0)e^{kx}$. From $\ln|y| = kx + c$ you get $y = Ae^{kx}$ with $A = \pm e^{c}$, and $A = 0$ is the constant solution, so A can be any real number.
- **Initial condition.** Use $y(x_0) = y_0$ to find C. Then check that the answer satisfies the equation.
- **Interval of validity.** A solution lives on the interval containing x_0 where the formula is defined and differentiable. If the formula blows up, the solution ends there; the formula's values on the far side belong to a different solution.
- **Newton's law of cooling.** $T' = -k(T - T_a)$ for room temperature T_a. The quantity that decays exponentially is $T - T_a$, not T.

Builds on: Integration: substitution and parts, Partial fractions and integrating rational functions

Problems

1. Solve $\dfrac{dy}{dx} = 3y$ with $y(0) = -2$. difficulty ·

2. Solve $\dfrac{dy}{dx} = \dfrac{x}{y}$ with $y(0) = 3$. difficulty ·

3. Solve $\dfrac{dy}{dx} = xy^2$ with $y(0) = 1$, and find the interval on which the solution exists. difficulty ··

4. Solve $\dfrac{dy}{dx} = e^{x-y}$ with $y(0) = \ln 2$. difficulty ··

5. A radioactive sample decays at a rate proportional to its mass, $\dfrac{dm}{dt} = -km$, with a half-life of 5 years. How long until 10% of it is left? difficulty ··

6. Coffee at 90°C is left in a 20°C room, and $T' = -k(T - 20)$. After 10 minutes it is at 60°C. When does it reach 30°C? difficulty ··

7. Solve $\dfrac{dy}{dx} = y(1 - y)$ with $y(0) = \tfrac{1}{2}$. difficulty ···

8. Gradient flow is gradient descent with an infinitely small step: $\dfrac{dw}{dt} = -\eta L'(w)$. For $L(w) = \tfrac{1}{2}(w - 3)^2$, $\eta = 0.1$ and $w(0) = 0$, find $w(t)$ and the time at which $w = 2.9$. difficulty ··

9. Solve $\dfrac{dy}{dx} = 2x(y - 1)$, first with $y(0) = 3$, then with $y(0) = 1$. difficulty ··

10. Water drains from a tank through a hole at the bottom, and its depth h (in meters, with t in minutes) satisfies $\dfrac{dh}{dt} = -\frac{1}{2}\sqrt{h}$, with $h(0) = 16$. When is the tank empty, and when is it half full? difficulty ···

Answers

1. $y = -2e^{3x}$
2. $y = \sqrt{x^2 + 9}$, for all x
3. $y = \dfrac{2}{2 - x^2}$, on $-\sqrt{2} < x < \sqrt{2}$
4. $y = \ln(1 + e^x)$, the softplus function
5. $\dfrac{5 \ln 10}{\ln 2} \approx 16.61$ years
6. About 34.77 minutes
7. $y = \dfrac{1}{1 + e^{-x}}$, the sigmoid
8. $w(t) = 3 - 3e^{-0.1t}$; $w = 2.9$ at $t = 10 \ln 30 \approx 34.01$
9. $y = 1 + 2e^{x^2}$ for $y(0) = 3$; $y \equiv 1$ for $y(0) = 1$
10. Empty at $t = 16$ minutes; half full at $t = 16 - 8\sqrt{2} \approx 4.69$ minutes

Worked solutions

Problem 1

> Solve $\frac{dy}{dx} = 3y$ with $y(0) = -2$.

$\frac{dy}{y} = 3\,dx$	Separate, for $y \neq 0$.
$\ln\lvert y\rvert = 3x + c$	Integrate both sides; one constant.
$\lvert y\rvert = e^{c}e^{3x}$, so $y = Ae^{3x}$ with $A = \pm e^{c}$	The absolute value is why A can be negative.
$y(0) = A = -2$	The initial condition.
$y = -2e^{3x}$	Check: $y' = -6e^{3x} = 3y$, and $y(0) = -2$. The solution stays negative and grows in size.

Problem 2

> Solve $\frac{dy}{dx} = \frac{x}{y}$ with $y(0) = 3$.

$y\,dy = x\,dx$	Multiply both sides by $y\,dx$.
$\frac{y^2}{2} = \frac{x^2}{2} + C$	Integrate both sides.
$C = \frac{9}{2}$, so $y^2 = x^2 + 9$	At $x = 0$, $y = 3$.
$y = \sqrt{x^2 + 9}$	Take the positive root, because $y(0) = 3 > 0$.
$y = \sqrt{x^2 + 9}$, **for all** x	Check: $y' = \frac{x}{\sqrt{x^2+9}} = \frac{x}{y}$. The curve is the upper branch of the hyperbola $y^2 - x^2 = 9$.

Problem 3

> Solve $\frac{dy}{dx} = xy^2$ with $y(0) = 1$, and find the interval on which the solution exists.

$y^{-2}\,dy = x\,dx$	Separate, for $y \neq 0$.
$-\frac{1}{y} = \frac{x^2}{2} + C$	$\int y^{-2}\,dy = -y^{-1}$.
$C = -1$	At $x = 0$: $-1 = 0 + C$.
$\frac{1}{y} = 1 - \frac{x^2}{2} = \frac{2 - x^2}{2}$, so $y = \frac{2}{2 - x^2}$	Solve for y only after C is in.
$y = \frac{2}{2 - x^2}$, **on** $-\sqrt{2} < x < \sqrt{2}$	The formula blows up at $x = \pm\sqrt{2} \approx \pm 1.414$, so the solution through $y(0) = 1$ exists only between them. Growth at a rate y^2 is faster than exponential and reaches infinity in finite time.

Problem 4

> Solve $\frac{dy}{dx} = e^{x-y}$ with $y(0) = \ln 2$.

$\frac{dy}{dx} = e^x e^{-y}$, so $e^y\,dy = e^x\,dx$	Split the exponential, then separate.
$e^y = e^x + C$	Integrate both sides.
$C = 1$	At $x = 0$: $e^{\ln 2} = 2 = 1 + C$.
$y = \ln(1 + e^x)$	Take the log; $1 + e^x > 0$, so this is defined for every x.
$y = \ln(1 + e^x)$, **the softplus function**	Check: $y' = \frac{e^x}{1+e^x}$ and $e^{x-y} = \frac{e^x}{1+e^x}$. The derivative of softplus is the sigmoid.

Problem 5

> A radioactive sample decays at a rate proportional to its mass, $\frac{dm}{dt} = -km$, with a half-life of 5 years. How long until 10% of it is left?

$m = m_0 e^{-kt}$	The solution of $m' = -km$, as in Problem 1.
$e^{-5k} = \frac{1}{2}$, so $k = \dfrac{\ln 2}{5}$	After one half-life, half the mass is left.
$e^{-kt} = \frac{1}{10}$, so $t = \dfrac{\ln 10}{k} = \dfrac{5 \ln 10}{\ln 2}$	Solve for the time when $m = 0.1 m_0$.
$\dfrac{5 \ln 10}{\ln 2} \approx 16.61$ **years**	About 3.32 half-lives: $2^{-3.32} \approx 0.1$. The initial mass cancels, so it was never needed.

Problem 6

> Coffee at $90°$C is left in a $20°$C room, and $T' = -k(T - 20)$. After 10 minutes it is at $60°$C. When does it reach $30°$C?

$\dfrac{dT}{T - 20} = -k\,dt$, so $T - 20 = Ae^{-kt}$	Separate and integrate; $T - 20$ decays exponentially.
$A = 90 - 20 = 70$	At $t = 0$.
$70e^{-10k} = 40$, so $k = \dfrac{\ln(7/4)}{10} \approx 0.0560$ per minute	At $t = 10$, $T - 20 = 40$.
$70e^{-kt} = 10$, so $t = \dfrac{\ln 7}{k} = \dfrac{10 \ln 7}{\ln(7/4)}$	At $T = 30$, $T - 20 = 10$.
About 34.77 **minutes**	T approaches $20°$C but never reaches it.

Problem 7

> Solve $\frac{dy}{dx} = y(1-y)$ with $y(0) = \frac{1}{2}$.

$\frac{dy}{y(1-y)} = dx$, and $\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}$	Separate, then partial fractions by the cover-up rule.
$\ln\lvert y\rvert - \ln\lvert 1-y\rvert = x + c$	$\int \frac{dy}{1-y} = -\ln\lvert 1-y\rvert$.
$\frac{y}{1-y} = Ae^x$, and $A = 1$	At $x = 0$, $\frac{1/2}{1/2} = 1$.
$y = e^x(1-y)$, so $y(1+e^x) = e^x$ and $y = \frac{e^x}{1+e^x}$	Solve for y.
$y = \frac{1}{1+e^{-x}}$, **the sigmoid**	Divide top and bottom by e^x. This is why $\sigma' = \sigma(1-\sigma)$, and why logistic growth with rate 1 and capacity 1 is a sigmoid. The constant solutions $y \equiv 0$ and $y \equiv 1$ are its two horizontal asymptotes.

Problem 8

> Gradient flow is gradient descent with an infinitely small step: $\frac{dw}{dt} = -\eta L'(w)$. For $L(w) = \frac{1}{2}(w-3)^2$, $\eta = 0.1$ and $w(0) = 0$, find $w(t)$ and the time at which $w = 2.9$.

$\frac{dw}{dt} = -0.1(w-3)$	$L'(w) = w - 3$.
$w - 3 = Ae^{-0.1t}$, and $A = -3$	The same structure as Problem 6: $w-3$ decays exponentially; $w(0) - 3 = -3$.
$w = 3 - 3e^{-0.1t}$	w climbs toward the minimizer 3.
$3e^{-0.1t} = 0.1$, so $t = 10\ln 30$	$w = 2.9$ means the gap is 0.1.

$w(t) = 3 - 3e^{-0.1t}$; $w = 2.9$ **at** $t = 10 \ln 30 \approx 34.01$	On a quadratic loss the distance to the minimum shrinks by the same factor in every unit of time. Gradient descent with step η shrinks it by $1 - \eta$ per step, and $(1 - \eta)^n \approx e^{-\eta n}$.

Problem 9

> Solve $\frac{dy}{dx} = 2x(y - 1)$, first with $y(0) = 3$, then with $y(0) = 1$.

$\frac{dy}{y - 1} = 2x\,dx$, for $y \neq 1$	Separate.
$\ln\lvert y - 1\rvert = x^2 + c$, so $y = 1 + Ae^{x^2}$	$A = \pm e^c$ is not 0 so far.
$y(0) = 3$ gives $A = 2$	$1 + A = 3$.
For $y(0) = 1$: $y \equiv 1$ satisfies $y' = 0 = 2x(1 - 1)$	The constant solution, which step 1 excluded by dividing by $y - 1$. It is the case $A = 0$.
$y = 1 + 2e^{x^2}$ **for** $y(0) = 3$; $y \equiv 1$ **for** $y(0) = 1$	Check: $y' = 4xe^{x^2} = 2x \cdot 2e^{x^2} = 2x(y - 1)$.

Problem 10

> Water drains from a tank through a hole at the bottom, and its depth h (in meters, with t in minutes) satisfies $\frac{dh}{dt} = -\frac{1}{2}\sqrt{h}$, with $h(0) = 16$. When is the tank empty, and when is it half full?

$h^{-1/2}\,dh = -\frac{1}{2}\,dt$	Separate, for $h > 0$.
$2\sqrt{h} = -\frac{t}{2} + C$, and $C = 8$	$\int h^{-1/2}\,dh = 2h^{1/2}$; at $t = 0$, $2 \cdot 4 = 8$.
$\sqrt{h} = 4 - \frac{t}{4}$, so $h = \left(4 - \frac{t}{4}\right)^2$ for $0 \le t \le 16$	Valid while $\sqrt{h} \ge 0$.

$h = 0$ at $t = 16$	After that $h \equiv 0$: the formula $\left(4 - \frac{t}{4}\right)^2$ would start rising again, which an empty tank does not do.
$\sqrt{8} = 4 - \frac{t}{4}$, so $t = 16 - 4\sqrt{8} = 16 - 8\sqrt{2}$	Half full means $h = 8$, if the tank has straight sides.
Empty at $t = 16$ minutes; half full at $t = 16 - 8\sqrt{2} \approx 4.69$ minutes	The first half drains in under a third of the time, because the flow is fastest when the water is deepest.

Where this goes wrong

1. Adding the constant after exponentiating

The $+C$ is often left until the end, as it would be for an ordinary antiderivative.

$\frac{dy}{y} = 3\,dx$, so $\ln\lvert y\rvert = 3x$	Right so far, apart from the constant: Problem 1, step 2.
"Solve for y, then add the constant."	The habit that causes the mistake: the constant belongs to the integration step, not to the final line.
$y = e^{3x} + C$, and $y(0) = -2$ gives $y = e^{3x} - 3$ ✗	Check it: $y' = 3e^{3x}$, but $3y = 3e^{3x} - 9$, so it does not satisfy the equation. Added at the integration step, $\ln\lvert y\rvert = 3x + c$ becomes $y = Ae^{3x}$, a multiplicative constant, and $y = -2e^{3x}$.

2. Splitting a square root

After solving for y^2, taking the square root looks like taking it term by term.

$y^2 = x^2 + 9$	Right so far: Problem 2, step 3.
"The square root of x^2 is x and of 9 is 3."	The habit that causes the mistake: $\sqrt{a+b}$ treated as $\sqrt{a} + \sqrt{b}$.

$y = x + 3$ ✗	At $x = 4$ this gives 7, but $\sqrt{16 + 9} = 5$. Check the equation too: $y' = 1$, while $\frac{x}{y} = \frac{x}{x+3}$, so $y = x + 3$ is not a solution. The answer is $y = \sqrt{x^2 + 9}$.

3. Losing the constant solution

Separating means dividing by the function of y, and that function can be 0.

$\frac{dy}{y-1} = 2x\,dx$, so $y = 1 + Ae^{x^2}$ with $A = \pm e^c$	Right so far: Problem 9, step 2.
"e^c is never 0, so $A \neq 0$."	The habit that causes the mistake: only the solutions the division allowed are kept.
With $y(0) = 1$, $1 + A = 1$ needs $A = 0$, so there is no solution ✗	$y \equiv 1$ is a solution: both sides of $y' = 2x(y-1)$ are 0. Dividing by $y - 1$ excluded it. Whenever you divide by $h(y)$, check the values with $h(y) = 0$ separately; here they make $A = 0$ allowed.

4. Forgetting the room temperature in Newton's law

Exponential decay is usually $y = y_0 e^{-kt}$, and that pattern gets applied to T itself.

$T' = -k(T - 20)$, $T(0) = 90$, $T(10) = 60$	Right so far: Problem 6.
"T decays exponentially, so $T = 90e^{-kt}$."	The habit that causes the mistake: the shift by the room temperature is dropped.
$90e^{-10k} = 60$ gives $k = \frac{\ln 1.5}{10}$, and $T = 30$ at $t = \frac{10 \ln 3}{\ln 1.5} \approx 27.10$ minutes ✗	$T = 90e^{-kt}$ decays to 0°C, not to the room's 20°C. It is $T - 20$ that decays: $70e^{-10k} = 40$, and the coffee reaches 30°C at about 34.77 minutes.

5. Using the formula past the blow-up

The formula for the solution is defined on both sides of the point where it blows up.

$y = \dfrac{2}{2 - x^2}$ solves $y' = xy^2$ with $y(0) = 1$	Right so far: Problem 3, step 4.
“Plug in $x = 2$.”	The habit that causes the mistake: the formula treated as valid wherever it can be evaluated.
$y(2) = \dfrac{2}{2 - 4} = -1$ ✗	The solution through $y(0) = 1$ goes to $+\infty$ at $x = \sqrt{2} \approx 1.414$ and stops there; a solution must be differentiable on an interval, and it cannot jump from $+\infty$ to negative values. The branch beyond $\sqrt{2}$ is a different solution, one that never passes through $(0, 1)$. $y(2)$ is not defined for this problem.

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