Second-order linear differential equations with constant coefficients

Ten second-order linear differential equations with constant coefficients: distinct real roots with an initial value problem, a repeated root and the extra factor of t, complex roots and a decaying oscillation, a spring's angular frequency, period and amplitude, underdamped, critically damped and overdamped springs, undetermined coefficients for a polynomial and an exponential input, the resonant case that needs t times the guess, a sinusoidal input with its steady-state amplitude and lag, superposition of two inputs, and the first-order system x′ = Ax whose eigenvalues are the characteristic roots, with worked solutions and the mistakes of giving a repeated root two copies of one exponential, flipping the sign of the decay, guessing a solution of the homogeneous equation and applying the initial conditions before the particular solution is added.

BEFORE YOU START

A second-order linear equation with constant coefficients, $ay'' + by' + cy = f(t)$, describes anything that has inertia and a restoring force: a mass on a spring, a circuit with an inductor and a capacitor, a pendulum swinging through small angles, and gradient descent with momentum on a quadratic loss. The homogeneous equation, with $f = 0$, is solved by exponentials e^{rt}, and putting one in turns the differential equation into a quadratic for r. Its two roots decide everything: two real roots give two exponentials, a repeated root needs an extra factor of t, and complex roots give oscillations inside an exponential envelope. A forcing term $f(t)$ adds one particular solution, found by guessing its form. The work goes wrong at a handful of places: the repeated root, the sign of the real part of complex roots, a guess that already solves the homogeneous equation, and initial conditions applied too early. These ten problems cover each of them.

- **Characteristic equation.** Putting $y = e^{rt}$ into $ay'' + by' + cy = 0$ gives $e^{rt}(ar^2 + br + c) = 0$, so r must solve $ar^2 + br + c = 0$.
- **Distinct real roots** $r_1 \neq r_2$: $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.
- **A repeated root** r (when $b^2 = 4ac$): $y = (c_1 + c_2 t)e^{rt}$. The second solution is te^{rt}, not a second copy of e^{rt}.
- **Complex roots** $r = \alpha \pm i\beta$: $y = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)$, from Euler's formula $e^{i\beta t} = \cos \beta t + i \sin \beta t$. The solution decays if $\alpha < 0$ and grows if $\alpha > 0$. β is an angular frequency in radians per unit time: the oscillation has period $\frac{2\pi}{\beta}$ and makes $\frac{\beta}{2\pi}$ cycles per unit time.
- **Initial conditions.** A second-order equation needs two: $y(0)$ and $y'(0)$. Two constants, two conditions.

- **Springs.** $mx'' + cx' + kx = 0$, with mass m, damping c and stiffness k. Without damping, $x'' + \omega^2 x = 0$ with $\omega = \sqrt{k/m}$, and $A\cos\omega t + B\sin\omega t$ has amplitude $\sqrt{A^2 + B^2}$. With damping, the discriminant $c^2 - 4mk$ decides: negative is **underdamped** (it oscillates), zero is **critically damped**, positive is **overdamped**.
- **Undetermined coefficients.** For $ay'' + by' + cy = f(t)$, $y = y_h + y_p$: the general homogeneous solution plus one particular solution. Guess y_p in the form of f: a polynomial of the same degree for a polynomial, Ae^{kt} for e^{kt}, $A\cos\omega t + B\sin\omega t$ for either a cosine or a sine. If the guess solves the homogeneous equation, multiply it by t (by t^2 for a repeated root). For a sum of inputs, add the particular solutions.
- **Constants last.** Apply the initial conditions to $y_h + y_p$, the full solution, never to y_h alone.

Problems

1. Solve $y'' + y' - 6y = 0$ with $y(0) = 1$ and $y'(0) = 7$. difficulty ·

2. Solve $y'' + 6y' + 9y = 0$ with $y(0) = 2$ and $y'(0) = -1$. difficulty ·

3. Solve $y'' + 2y' + 5y = 0$ with $y(0) = 1$ and $y'(0) = 3$. difficulty ··

4. A 2 kg mass hangs on a spring of stiffness 50 N/m with no damping, so $2x'' + 50x = 0$. It starts 0.1 m from equilibrium with velocity 1 m/s. Find $x(t)$, its amplitude, its period and its frequency in cycles per second. difficulty ··

5. A spring with $m = 1$ and $k = 4$ has damping c, so $x'' + cx' + 4x = 0$. Classify the motion for $c = 1$, $c = 4$ and $c = 5$, find the critical value of c, and solve the critically damped case with $x(0) = 1$ and $x'(0) = 0$. difficulty ··

6. Solve $y'' + y' - 2y = 4t^2$ with $y(0) = 0$ and $y'(0) = 0$. difficulty ··

7. Find a particular solution of $y'' - y' - 6y = 12e^{t}$, and then of $y'' - y' - 6y = 10e^{3t}$. difficulty ··

8. A damped spring is driven by a periodic force: $x'' + 2x' + 5x = 10\cos t$. Find the steady-state solution, its amplitude and how far it lags the force, and find the driving frequency ω that makes the steady-state amplitude of $x'' + 2x' + 5x = 10\cos \omega t$ largest. difficulty ···

9. Solve $y'' + 4y = 8t + 5e^{-t}$ with $y(0) = 1$ and $y'(0) = 0$. difficulty ··

10. Write $y'' + 3y' + 2y = 0$ as a first-order system $\mathbf{x}' = A\mathbf{x}$ for $\mathbf{x} = (y, y')$. Show that the eigenvalues of A are the characteristic roots, find the eigenvectors, and solve with $y(0) = 1$, $y'(0) = 0$. difficulty ···

Answers

1. $y = 2e^{2t} - e^{-3t}$
2. $y = (2 + 5t)e^{-3t}$
3. $y = e^{-t}(\cos 2t + 2\sin 2t)$
4. $x = 0.1\cos 5t + 0.2\sin 5t$; amplitude ≈ 0.224 m, period $\frac{2\pi}{5} \approx 1.257$ s, frequency $\frac{5}{2\pi} \approx 0.796$ cycles per second
5. Underdamped for $c = 1$, critically damped for $c = 4$, overdamped for $c = 5$; $x = (1 + 2t)e^{-2t}$ at $c = 4$
6. $y = \frac{8}{3}e^{t} + \frac{1}{3}e^{-2t} - 2t^2 - 2t - 3$
7. $y_p = -2e^{t}$ for $12e^{t}$, and $y_p = 2te^{3t}$ for $10e^{3t}$
8. $x_p = 2\cos t + \sin t$, amplitude $\sqrt{5}$, lag ≈ 0.464 rad; the amplitude is largest at $\omega = \sqrt{3}$, where it is 2.5
9. $y = 2t + e^{-t} - \frac{1}{2}\sin 2t$
10. $y = 2e^{-t} - e^{-2t}$

Worked solutions

Problem 1

> Solve $y'' + y' - 6y = 0$ with $y(0) = 1$ and $y'(0) = 7$.

$r^2 + r - 6 = (r+3)(r-2) = 0$, so $r = 2$ or $r = -3$	Put $y = e^{rt}$ in and divide by e^{rt}, which is never 0.
$y = c_1 e^{2t} + c_2 e^{-3t}$	Two distinct real roots, two exponentials.
$y(0) = c_1 + c_2 = 1$ and $y'(0) = 2c_1 - 3c_2 = 7$	Differentiate before substituting $t = 0$.
$2c_1 - 3(1 - c_1) = 7$, so $5c_1 = 10$ and $c_1 = 2$, $c_2 = -1$	Substitute $c_2 = 1 - c_1$.
$y = 2e^{2t} - e^{-3t}$	Check: $y'' + y' - 6y = (8 + 4 - 12)e^{2t} + (-9 + 3 + 6)e^{-3t} = 0$, and $y'(0) = 4 + 3 = 7$. The e^{2t} term takes over for large t.

Problem 2

> Solve $y'' + 6y' + 9y = 0$ with $y(0) = 2$ and $y'(0) = -1$.

$r^2 + 6r + 9 = (r+3)^2 = 0$, so $r = -3$ twice	The discriminant $36 - 36$ is 0.
$y = (c_1 + c_2 t)e^{-3t}$	A repeated root gives e^{-3t} and te^{-3t}. Check the second: $(te^{-3t})'' + 6(te^{-3t})' + 9te^{-3t} = (9t - 6) + 6(1 - 3t) + 9t = 0$, times e^{-3t}.
$y(0) = c_1 = 2$	$t = 0$ removes the c_2 term.

$y' = \big(c_2 - 3(c_1 + c_2 t)\big)e^{-3t}$, so $y'(0) = c_2 - 3c_1 = -1$ and $c_2 = 5$	The product rule on $(c_1 + c_2 t)e^{-3t}$.
$y = (2 + 5t)e^{-3t}$	Check: $y' = (-1 - 15t)e^{-3t}$, so $y'(0) = -1$, and $y'' + 6y' + 9y = 0$. One exponential alone could not meet both conditions (Mistake 1).

Problem 3

> Solve $y'' + 2y' + 5y = 0$ with $y(0) = 1$ and $y'(0) = 3$.

$r = \dfrac{-2 \pm \sqrt{4 - 20}}{2} = \dfrac{-2 \pm 4i}{2} = -1 \pm 2i$	The discriminant is -16, so the roots are complex.
$y = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$	$\alpha = -1$ is the real part and $\beta = 2$ the imaginary part.
$y(0) = c_1 = 1$	$\cos 0 = 1$ and $\sin 0 = 0$.
$y' = e^{-t}\big((2c_2 - c_1)\cos 2t - (2c_1 + c_2)\sin 2t\big)$, so $y'(0) = 2c_2 - c_1 = 3$ and $c_2 = 2$	The product rule, then $t = 0$.
$y = e^{-t}(\cos 2t + 2 \sin 2t)$	An oscillation with period $\frac{2\pi}{2} = \pi$ inside the decaying envelope $\sqrt{5}\,e^{-t}$, where $\sqrt{1^2 + 2^2} = \sqrt{5}$.

Problem 4

> A 2 kg mass hangs on a spring of stiffness 50 N/m with no damping, so $2x'' + 50x = 0$. It starts 0.1 m from equilibrium with velocity 1 m/s. Find $x(t)$, its amplitude, its period and its frequency in cycles per second.

$x'' + 25x = 0$, so $r^2 + 25 = 0$ and $r = \pm 5i$	Divide by the mass. $\omega = \sqrt{k/m} = \sqrt{50/2} = 5$ rad/s.
$x = A\cos 5t + B \sin 5t$	$\alpha = 0$: no decay.

$x(0) = A = 0.1$ and $x'(0) = 5B = 1$, so $B = 0.2$	$x' = -5A\sin 5t + 5B\cos 5t$.
Amplitude $\sqrt{0.1^2 + 0.2^2} = \sqrt{0.05} \approx 0.2236$ m	$A\cos\omega t + B\sin\omega t = R\cos(\omega t - \varphi)$ with $R = \sqrt{A^2 + B^2}$.
$x = 0.1\cos 5t + 0.2\sin 5t$; **amplitude** ≈ 0.224 **m, period** $\frac{2\pi}{5} \approx 1.257$ **s, frequency** $\frac{5}{2\pi} \approx 0.796$ **cycles per second**	Check: $x\left(\frac{2\pi}{5}\right) = 0.1$, back where it started. The period depends only on m and k, not on how far the spring is pulled.

Problem 5

> A spring with $m = 1$ and $k = 4$ has damping c, so $x'' + cx' + 4x = 0$. Classify the motion for $c = 1$, $c = 4$ and $c = 5$, find the critical value of c, and solve the critically damped case with $x(0) = 1$ and $x'(0) = 0$.

$r^2 + cr + 4 = 0$ has discriminant $c^2 - 16$	Critical damping is where it is 0: $c = 2\sqrt{mk} = 4$.
$c = 1$: $r = \dfrac{-1 \pm i\sqrt{15}}{2}$, underdamped	It oscillates at $\beta = \frac{\sqrt{15}}{2} \approx 1.936$ rad/s, a little slower than the undamped 2, inside the envelope $e^{-t/2}$.
$c = 5$: $r^2 + 5r + 4 = (r + 1)(r + 4)$, so $r = -1, -4$, overdamped	No oscillation; the slow mode e^{-t} sets how fast it settles.
$c = 4$: $r = -2$ twice, so $x = (c_1 + c_2 t)e^{-2t}$ with $c_1 = 1$ and $c_2 - 2c_1 = 0$	$x'(0) = c_2 - 2c_1$, as in Problem 2.
Underdamped for $c = 1$, **critically damped for** $c = 4$, **overdamped for** $c = 5$; $x = (1 + 2t)e^{-2t}$ **at** $c = 4$	The critically damped spring settles fastest without overshooting, as e^{-2t}: more damping (e^{-t} for $c = 5$) or less ($e^{-t/2}$ for $c = 1$) is slower.

Problem 6

> Solve $y'' + y' - 2y = 4t^2$ with $y(0) = 0$ and $y'(0) = 0$.

$r^2 + r - 2 = (r+2)(r-1)$, so $y_h = c_1 e^{t} + c_2 e^{-2t}$	The homogeneous equation first.
Guess $y_p = At^2 + Bt + C$, so $y_p' = 2At + B$ and $y_p'' = 2A$	A quadratic input needs a full quadratic guess, lower terms included.
$2A + (2At + B) - 2(At^2 + Bt + C) = -2At^2 + (2A - 2B)t + (2A + B - 2C) = 4t^2$	Collect powers of t.
$-2A = 4$, $2A - 2B = 0$, $2A + B - 2C = 0$ give $A = -2$, $B = -2$, $C = -3$	Match the coefficients of t^2, t and 1.
$y = \frac{8}{3}e^{t} + \frac{1}{3}e^{-2t} - 2t^2 - 2t - 3$	The conditions go on the full solution: $c_1 + c_2 - 3 = 0$ and $c_1 - 2c_2 - 2 = 0$, so $c_1 = \frac{8}{3}$, $c_2 = \frac{1}{3}$. Check: $y(0) = \frac{8}{3} + \frac{1}{3} - 3 = 0$.

Problem 7

> Find a particular solution of $y'' - y' - 6y = 12e^{t}$, and then of $y'' - y' - 6y = 10e^{3t}$.

$r^2 - r - 6 = (r-3)(r+2)$, so $y_h = c_1 e^{3t} + c_2 e^{-2t}$	Find the homogeneous solutions before guessing.
$y_p = Ae^{t}$: $A(1 - 1 - 6)e^{t} = -6Ae^{t} = 12e^{t}$, so $A = -2$	e^{t} is not a homogeneous solution, so the plain guess works.
Ae^{3t} fails for the second input: it gives $A(9 - 3 - 6)e^{3t} = 0$	e^{3t} already solves the homogeneous equation (Mistake 3).

$y_p = Ate^{3t}$: $y_p' = A(1+3t)e^{3t}$, $y_p'' = A(6+9t)e^{3t}$, and $A\big((6+9t)-(1+3t)-6t\big)e^{3t} = 5Ae^{3t} = 10e^{3t}$	The t terms cancel, as they must; $A = 2$.
$y_p = -2e^t$ **for** $12e^t$**, and** $y_p = 2te^{3t}$ **for** $10e^{3t}$	The general solution of the second is $c_1e^{3t} + c_2e^{-2t} + 2te^{3t}$.

Problem 8

> A damped spring is driven by a periodic force: $x'' + 2x' + 5x = 10\cos t$. Find the steady-state solution, its amplitude and how far it lags the force, and find the driving frequency ω that makes the steady-state amplitude of $x'' + 2x' + 5x = 10\cos \omega t$ largest.

$x_p = A\cos t + B\sin t$, $x_p' = -A\sin t + B\cos t$, $x_p'' = -A\cos t - B\sin t$	A cosine input needs both a cosine and a sine in the guess.
Cosines: $-A + 2B + 5A = 4A + 2B = 10$; sines: $-B - 2A + 5B = 4B - 2A = 0$	Collect $\cos t$ and $\sin t$ separately.
$A = 2B$, so $10B = 10$: $B = 1$ and $A = 2$	The homogeneous part $e^{-t}(c_1\cos 2t + c_2\sin 2t)$ dies out, so x_p is what remains.
Amplitude $\sqrt{2^2+1^2} = \sqrt{5} \approx 2.236$, lag $\varphi = \arctan\frac{1}{2} \approx 0.464$ rad	$2\cos t + \sin t = \sqrt{5}\cos(t-\varphi)$. The same comes from $\dfrac{10}{\sqrt{(5-\omega^2)^2 + (2\omega)^2}} = \dfrac{10}{\sqrt{20}}$ at $\omega = 1$.
$x_p = 2\cos t + \sin t$**, amplitude** $\sqrt{5}$**, lag** ≈ 0.464 **rad; the amplitude is largest at** $\omega = \sqrt{3}$**, where it is** 2.5	Minimize $(5-\omega^2)^2 + 4\omega^2 = \omega^4 - 6\omega^2 + 25$: its derivative in ω^2 is $2\omega^2 - 6$, zero at $\omega^2 = 3$, where it equals 16 and the amplitude is $\frac{10}{4}$. Damping moves the peak below the natural frequency $\sqrt{5}$ and keeps it finite.

Problem 9

> Solve $y'' + 4y = 8t + 5e^{-t}$ with $y(0) = 1$ and $y'(0) = 0$.

$r^2 + 4 = 0$, so $y_h = c_1 \cos 2t + c_2 \sin 2t$	$r = \pm 2i$.
For $8t$: $y_1 = at + b$ gives $4at + 4b = 8t$, so $y_1 = 2t$	$y_1'' = 0$.
For $5e^{-t}$: $y_2 = Ce^{-t}$ gives $(1 + 4)Ce^{-t} = 5e^{-t}$, so $y_2 = e^{-t}$	Superposition: solve for each input separately and add.
$y(0) = c_1 + 1 = 1$ and $y'(0) = 2c_2 + 2 - 1 = 0$, so $c_1 = 0$ and $c_2 = -\frac{1}{2}$	$y' = -2c_1 \sin 2t + 2c_2 \cos 2t + 2 - e^{-t}$.
$y = 2t + e^{-t} - \frac{1}{2} \sin 2t$	Check: $y'' = e^{-t} + 2 \sin 2t$ and $4y = 8t + 4e^{-t} - 2 \sin 2t$ add to $8t + 5e^{-t}$.

Problem 10

> Write $y'' + 3y' + 2y = 0$ as a first-order system $\mathbf{x}' = A\mathbf{x}$ for $\mathbf{x} = (y, y')$. Show that the eigenvalues of A are the characteristic roots, find the eigenvectors, and solve with $y(0) = 1$, $y'(0) = 0$.

$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$	The first row says $(y)' = y'$; the second says $(y')' = -2y - 3y'$.
$\det(A - rI) = (-r)(-3 - r) + 2 = r^2 + 3r + 2$	The characteristic polynomial of A is the characteristic polynomial of the equation, so the eigenvalues are $r = -1$ and $r = -2$.
Eigenvectors $(1, -1)$ for $r = -1$ and $(1, -2)$ for $r = -2$	The first row of $(A - rI)\mathbf{v} = 0$ is $-rv_1 + v_2 = 0$, so $\mathbf{v} = (1, r)$: a solution e^{rt} and its derivative re^{rt}.
$\mathbf{x} = c_1 e^{-t}(1, -1) + c_2 e^{-2t}(1, -2)$ with $c_1 + c_2 = 1$ and $-c_1 - 2c_2 = 0$	So $c_1 = 2$ and $c_2 = -1$.

$y = 2e^{-t} - e^{-2t}$	Every equation here is a 2×2 system in disguise. Gradient descent with momentum on a quadratic of curvature λ behaves like $y'' + \gamma y' + \lambda y = 0$, and it converges fastest at the critical damping $\gamma = 2\sqrt{\lambda}$, as in Problem 5.

Where this goes wrong

1. A repeated root written as two copies of one exponential

The rule for distinct roots, one exponential per root, is applied to a double root.

$r^2 + 6r + 9 = (r+3)^2$, so $r = -3$ twice	Right so far: Problem 2.
"Two roots, so two exponentials."	The step that causes the mistake: both roots give the same function.
$y = c_1 e^{-3t} + c_2 e^{-3t}$ ✗	This is $(c_1 + c_2)e^{-3t}$, a single constant times one function. $y(0) = 2$ forces $Ce^{-3t} = 2e^{-3t}$, whose $y'(0)$ is -6, not the required -1; no choice fits both conditions. The second solution is te^{-3t}, and $y = (2 + 5t)e^{-3t}$.

2. The sign of the real part of complex roots flipped

The roots are complex, and the decay rate is read off the equation's coefficient instead of the roots.

$y'' + 2y' + 5y = 0$ has $r = -1 \pm 2i$	Right so far: Problem 3.
"The coefficient of y' is $+2$, so $\alpha = +1$."	The step that causes the mistake: α is the real part of the root, $-\frac{b}{2a} = -1$.

$y = e^t(\cos 2t + \sin 2t)$ ✗	With $y(0) = 1$ and $y'(0) = 1 + 2c_2 = 3$, this meets both conditions but grows instead of decaying. It solves $y'' - 2y' + 5y = 0$; in the original equation the left side is $4y'$, which is 12 at $t = 0$, not 0. A positive damping coefficient always means decay: $y = e^{-t}(\cos 2t + 2\sin 2t)$.

3. Guessing a solution of the homogeneous equation

The input is e^{3t}, so the guess is e^{3t}, as it was for e^t.

$y'' - y' - 6y = 10e^{3t}$, with characteristic roots 3 and -2	Right so far: Problem 7.
Guess $y_p = Ae^{3t}$	The step that causes the mistake: e^{3t} is already in y_h.
$A(9 - 3 - 6)e^{3t} = 10e^{3t}$, so $0 = 10$ and the equation has no particular solution ✗	The equation has one; the guess was in the wrong family. Anything that solves the homogeneous equation gives 0 on the left. Multiply the guess by t: $y_p = 2te^{3t}$.

4. Initial conditions applied before adding the particular solution

The constants are found from y_h, and the particular solution is added afterward.

$y_h = c_1e^t + c_2e^{-2t}$ and $y_p = -2t^2 - 2t - 3$	Right so far: Problem 6.
$c_1 + c_2 = 0$ and $c_1 - 2c_2 = 0$ from $y(0) = 0$ and $y'(0) = 0$	The step that causes the mistake: these conditions belong to y, not to y_h.
$c_1 = c_2 = 0$, so $y = -2t^2 - 2t - 3$ ✗	This solves the differential equation but has $y(0) = -3$ and $y'(0) = -2$, not 0 and 0. Apply both conditions to $y_h + y_p$: $c_1 = \frac{8}{3}$ and $c_2 = \frac{1}{3}$.

5. The period taken as β/2π

$\omega = 5$ rad/s is right, and the period is written down from it the wrong way up.

$x = 0.1\cos 5t + 0.2\sin 5t$	Right so far: Problem 4.
"The period is ω over 2π."	The step that causes the mistake: $\frac{\omega}{2\pi}$ is the frequency, in cycles per second.
Period $= \frac{5}{2\pi} \approx 0.796$ s ✗	After 0.796 s the mass is at $x \approx -0.216$ m, near the far end of its swing, not back at 0.1. One full cycle is $5t = 2\pi$, so the period is $\frac{2\pi}{5} \approx 1.257$ s; check the units, since ω is in radians per second.

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