Riemann sums: left, right, midpoint and trapezoid, and the limit that defines the integral

Ten problems on Riemann sums: left and right sums for x² on [0, 2], the midpoint sum and its error, the trapezoid rule and Simpson's combination, over- and underestimates for 1/x, the limit of right sums for 2x + 1 and for x², a sum recognized as π/4, distance from a table of speeds, how many subintervals an error bound needs, and the area under an ROC curve with uneven steps, with worked solutions and the mistakes of using every point in a left sum, writing x_i = i/n on the wrong interval and assuming equal widths.

BEFORE YOU START

A definite integral is defined as a limit of sums: cut the interval into thin strips, approximate each strip by a rectangle, add up the rectangles, and let the strips get thinner. Those sums are also how integrals are computed when there is no antiderivative, only a table of measurements or a curve known at a few points, as with the area under an ROC curve. The choice of where each rectangle touches the curve decides whether the sum is too big or too small, and how fast the error shrinks. The work goes wrong in a few places: a left sum that uses every point, x_i written as if every interval were $[0, 1]$, the endpoint weights of the trapezoid rule forgotten, equal widths assumed for unequal data, and the belief that a left sum always underestimates. These ten problems cover each of them.

- **Partition.** Split $[a, b]$ into n equal pieces of width $\Delta x = \dfrac{b-a}{n}$, with points $x_i = a + i\,\Delta x$ for $i = 0, 1, \dots, n$. There are n strips and $n+1$ points.
- **Left, right, midpoint.** $L_n = \Delta x \displaystyle\sum_{i=0}^{n-1} f(x_i)$, $R_n = \Delta x \displaystyle\sum_{i=1}^{n} f(x_i)$, and $M_n = \Delta x \displaystyle\sum_{i=1}^{n} f\Big(\frac{x_{i-1}+x_i}{2}\Big)$. Each uses exactly n heights.
- **Trapezoid and Simpson.** $T_n = \dfrac{\Delta x}{2}\big(f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\big) = \dfrac{L_n + R_n}{2}$. With unequal widths, add $\frac{1}{2}(x_{i+1} - x_i)\big(f(x_i) + f(x_{i+1})\big)$ strip by strip. $\dfrac{2M_n + T_n}{3}$ is Simpson's rule on $2n$ strips.
- **Over or under.** If f is increasing, L_n is too small and R_n too big; if decreasing, the other way round. If f is concave up, M_n is too small and T_n too big.

- **The definition.** $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x$. Useful sums: $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.
- **Error bounds.** If $|f''| \le K$ on $[a,b]$, then $|E_M| \le \dfrac{K(b-a)^3}{24n^2}$ and $|E_T| \le \dfrac{K(b-a)^3}{12n^2}$.

Builds on: Evaluating limits: factoring, conjugates and limits at infinity

Problems

1. Find the left and right sums L_4 and R_4 for $f(x) = x^2$ on $[0,2]$, and compare them with the integral. difficulty ·

2. Find the midpoint sum M_4 for x^2 on $[0,2]$, and its error. difficulty ·

3. Find the trapezoid sum T_4 for x^2 on $[0,2]$ and its error, and then the combination $\dfrac{2M_4 + T_4}{3}$. difficulty ··

4. Find L_4 and R_4 for $f(x) = \dfrac{1}{x}$ on $[1,2]$. Which is an overestimate? difficulty ··

5. Write $\displaystyle\int_0^3 (2x+1)\,dx$ as a limit of right sums and evaluate the limit. difficulty ··

6. Use right sums to show that $\displaystyle\int_0^1 x^2\,dx = \frac{1}{3}$, and find R_{10}. difficulty ···

7. Evaluate $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+i^2}$ by recognizing it as a Riemann sum. difficulty ··

8. A car's speed is recorded every 10 seconds: 0, 8, 14, 18 and 20 m/s at $t = 0, 10, 20, 30, 40$. Estimate the distance traveled with left, right and trapezoid sums. difficulty ··

9. How many subintervals guarantee an error below 10^{-4} for $\int_1^2 \frac{dx}{x}$ with the midpoint rule, and with the trapezoid rule? difficulty ···

10. A classifier's ROC curve passes through the points (false positive rate, true positive rate) $(0, 0)$, $(0.1, 0.5)$, $(0.3, 0.8)$, $(0.6, 0.9)$ and $(1, 1)$, joined by straight lines. Find the area under the curve. difficulty ···

Answers

1. $L_4 = 1.75$, $R_4 = 3.75$; the integral $\frac{8}{3}$ lies between them
2. $M_4 = 2.625$, an error of $\frac{8}{3} - \frac{21}{8} = \frac{1}{24} \approx 0.0417$
3. $T_4 = 2.75$ with error $-\frac{1}{12}$; $\frac{2M_4 + T_4}{3} = \frac{8}{3}$ exactly
4. $L_4 \approx 0.7595$ is the overestimate and $R_4 \approx 0.6345$ the underestimate
5. 12
6. $\displaystyle\int_0^1 x^2\,dx = \frac{1}{3}$; $R_{10} = \frac{11 \cdot 21}{600} = 0.385$
7. $\frac{\pi}{4} \approx 0.7854$
8. $L_4 = 400$ m, $R_4 = 600$ m, $T_4 = 500$ m
9. Midpoint: $n = 29$; trapezoid: $n = 41$
10. $\text{AUC} = 0.025 + 0.13 + 0.255 + 0.38 = 0.79$

Worked solutions

Problem 1

> Find the left and right sums L_4 and R_4 for $f(x) = x^2$ on $[0, 2]$, and compare them with the integral.

$\Delta x = \frac{2-0}{4} = \frac{1}{2}$; points $0, \frac{1}{2}, 1, \frac{3}{2}, 2$; heights $0, \frac{1}{4}, 1, \frac{9}{4}, 4$	Five points, four strips.
$L_4 = \frac{1}{2}\left(0 + \frac{1}{4} + 1 + \frac{9}{4}\right) = \frac{1}{2} \cdot \frac{7}{2} = \frac{7}{4}$	The first four heights.
$R_4 = \frac{1}{2}\left(\frac{1}{4} + 1 + \frac{9}{4} + 4\right) = \frac{1}{2} \cdot \frac{15}{2} = \frac{15}{4}$	The last four heights.
$\displaystyle\int_0^2 x^2\,dx = \frac{8}{3} \approx 2.667$	The exact value, for comparison.
$L_4 = 1.75$, $R_4 = 3.75$; **the integral $\frac{8}{3}$ lies between them**	x^2 is increasing on $[0, 2]$, so left rectangles sit under the curve and right ones stick out above it. $R_4 - L_4 = \Delta x\,\big(f(2) - f(0)\big) = 2$.

Problem 2

> Find the midpoint sum M_4 for x^2 on $[0, 2]$, and its error.

Midpoints $\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}$	Halfway along each strip of width $\frac{1}{2}$.
Heights $\frac{1}{16}, \frac{9}{16}, \frac{25}{16}, \frac{49}{16}$, which add to $\frac{84}{16} = \frac{21}{4}$	Square each midpoint.
$M_4 = \frac{1}{2} \cdot \frac{21}{4} = \frac{21}{8}$	Times Δx.
$M_4 = 2.625$, **an error of** $\frac{8}{3} - \frac{21}{8} = \frac{1}{24} \approx 0.0417$	With the same four heights' worth of work, it is much closer than L_4 or R_4, and slightly low because x^2 is concave up.

Problem 3

> Find the trapezoid sum T_4 for x^2 on $[0, 2]$ and its error, and then the combination $\dfrac{2M_4 + T_4}{3}$.

$T_4 = \frac{\Delta x}{2}\big(f(0) + 2f(\frac{1}{2}) + 2f(1) + 2f(\frac{3}{2}) + f(2)\big) = \frac{1}{4}\big(0 + \frac{1}{2} + 2 + \frac{9}{2} + 4\big)$	Interior points count twice, the ends once.
$T_4 = \frac{1}{4} \cdot 11 = \frac{11}{4} = 2.75$, which is also $\frac{L_4 + R_4}{2}$	The average of 1.75 and 3.75.
Error $\frac{8}{3} - \frac{11}{4} = -\frac{1}{12}$	Too big, because the chords of a concave-up curve lie above it. It is -2 times the midpoint error $\frac{1}{24}$.
$\dfrac{2 \cdot \frac{21}{8} + \frac{11}{4}}{3} = \dfrac{\frac{21}{4} + \frac{11}{4}}{3} = \dfrac{8}{3}$	The weights 2 and 1 cancel the two errors.
$T_4 = 2.75$ **with error** $-\frac{1}{12}$**;** $\frac{2M_4 + T_4}{3} = \frac{8}{3}$ **exactly**	This combination is Simpson's rule, which is exact for every polynomial of degree 3 or less.

Problem 4

> Find L_4 and R_4 for $f(x) = \dfrac{1}{x}$ on $[1, 2]$. Which is an overestimate?

$\Delta x = \frac{1}{4}$; points $1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2$; heights $1, \frac{4}{5}, \frac{2}{3}, \frac{4}{7}, \frac{1}{2}$	$\frac{1}{5/4} = \frac{4}{5}$, and so on.
$L_4 = \frac{1}{4}\big(1 + \frac{4}{5} + \frac{2}{3} + \frac{4}{7}\big) \approx \frac{1}{4} \cdot 3.0381 \approx 0.7595$	The first four heights.
$R_4 = L_4 - \frac{1}{4}\big(1 - \frac{1}{2}\big) \approx 0.6345$	The same sum with $f(1)$ swapped for $f(2)$.
$\displaystyle\int_1^2 \frac{dx}{x} = \ln 2 \approx 0.6931$	The exact value.

$L_4 \approx 0.7595$ **is the overestimate and** $R_4 \approx 0.6345$ **the underestimate**	$\frac{1}{x}$ is decreasing, so each strip's left edge is its highest point.

Problem 5

> Write $\int_0^3 (2x+1)\,dx$ as a limit of right sums and evaluate the limit.

$\Delta x = \frac{3}{n}$ and $x_i = 0 + i \cdot \frac{3}{n} = \frac{3i}{n}$	Equal widths on $[0, 3]$.
$R_n = \sum_{i=1}^{n} \left(2 \cdot \frac{3i}{n} + 1\right)\frac{3}{n} = \frac{18}{n^2}\sum_{i=1}^{n} i + \frac{3}{n}\sum_{i=1}^{n} 1$	Split the sum and pull out what does not depend on i.
$= \frac{18}{n^2} \cdot \frac{n(n+1)}{2} + 3 = \frac{9(n+1)}{n} + 3 = 12 + \frac{9}{n}$	$\sum_{i=1}^{n} 1 = n$.
$12 + \frac{9}{n} \to 12$	Take the limit.
12	Check with an antiderivative: $\left[x^2 + x\right]_0^3 = 12$. For an increasing function every R_n is above the integral, here by exactly $\frac{9}{n}$.

Problem 6

> Use right sums to show that $\int_0^1 x^2\,dx = \frac{1}{3}$, and find R_{10}.

$\Delta x = \frac{1}{n}$, $x_i = \frac{i}{n}$, so $R_n = \sum_{i=1}^{n} \frac{i^2}{n^2} \cdot \frac{1}{n} = \frac{1}{n^3}\sum_{i=1}^{n} i^2$	Right endpoints on $[0, 1]$.

$R_n = \dfrac{1}{n^3}\cdot\dfrac{n(n+1)(2n+1)}{6} = \dfrac{(n+1)(2n+1)}{6n^2}$	The sum of squares.
$\dfrac{(n+1)(2n+1)}{6n^2} = \dfrac{2n^2+3n+1}{6n^2} \to \dfrac{2}{6} = \dfrac{1}{3}$	Divide top and bottom by n^2.
$\displaystyle\int_0^1 x^2\,dx = \frac{1}{3};\ R_{10} = \tfrac{11\cdot 21}{600} = 0.385$	This is the calculation Archimedes did for the parabola, with no antiderivative in sight. R_{10} is still 0.052 too high.

Problem 7

Evaluate $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+i^2}$ by recognizing it as a Riemann sum.

$\dfrac{n}{n^2+i^2} = \dfrac{1}{n}\cdot\dfrac{1}{1+(i/n)^2}$	Divide top and bottom by n^2 to expose $\frac{i}{n}$ and a factor $\frac{1}{n}$.
This is $\displaystyle\sum_{i=1}^{n} f(x_i)\,\Delta x$ with $f(x)=\frac{1}{1+x^2}$, $x_i=\frac{i}{n}$, $\Delta x=\frac{1}{n}$	A right sum on $[0,1]$.
$\displaystyle\int_0^1 \frac{dx}{1+x^2} = \arctan 1 - \arctan 0$	The limit of the sums is the integral.
$\frac{\pi}{4} \approx 0.7854$	At $n=1000$ the sum is about 0.78515, slightly low because $\frac{1}{1+x^2}$ is decreasing.

Problem 8

A car's speed is recorded every 10 seconds: 0, 8, 14, 18 and 20 m/s at $t = 0, 10, 20, 30, 40$. Estimate the distance traveled with left, right and trapezoid sums.

Distance is $\displaystyle\int_0^{40} v(t)\,dt$, and $\Delta t = 10$	Four strips of 10 s.

$L_4 = 10(0 + 8 + 14 + 18) = 400$ m	The speed at the start of each interval.
$R_4 = 10(8 + 14 + 18 + 20) = 600$ m	The speed at the end of each interval.
$T_4 = \frac{10}{2}\big(0 + 2 \cdot 8 + 2 \cdot 14 + 2 \cdot 18 + 20\big) = 5 \cdot 100 = 500$ m	The average of L_4 and R_4.
$L_4 = 400$ **m,** $R_4 = 600$ **m,** $T_4 = 500$ **m**	The car speeds up throughout, so the true distance is between 400 and 600 m, and the trapezoid estimate is the best of the three.

Problem 9

> How many subintervals guarantee an error below 10^{-4} for $\displaystyle\int_1^2 \frac{dx}{x}$ with the midpoint rule, and with the trapezoid rule?

$f''(x) = \dfrac{2}{x^3} \le 2$ on $[1, 2]$, so $K = 2$	f'' is largest at the left end.
Midpoint: $\dfrac{2 \cdot 1^3}{24n^2} = \dfrac{1}{12n^2} < 10^{-4}$ needs $n^2 > 833.3$	$28^2 = 784$ is too small and $29^2 = 841$ is enough.
Trapezoid: $\dfrac{2 \cdot 1^3}{12n^2} = \dfrac{1}{6n^2} < 10^{-4}$ needs $n^2 > 1666.7$	$40^2 = 1600$ is too small and $41^2 = 1681$ is enough.
Midpoint: $n = 29$; **trapezoid:** $n = 41$	The midpoint bound is half the trapezoid bound, so the trapezoid rule needs about $\sqrt{2}$ times as many strips. These are guarantees; the actual errors at these n are smaller still.

Problem 10

> A classifier's ROC curve passes through the points (false positive rate, true positive rate) $(0, 0)$, $(0.1, 0.5)$, $(0.3, 0.8)$, $(0.6, 0.9)$ and $(1, 1)$, joined by straight lines. Find the area under the curve.

The widths are 0.1, 0.2, 0.3 and 0.4	The steps in the false positive rate are not equal.
$\frac{1}{2} \cdot 0.1 \cdot (0 + 0.5) = 0.025$ and $\frac{1}{2} \cdot 0.2 \cdot (0.5 + 0.8) = 0.13$	One trapezoid per segment, each with its own width.
$\frac{1}{2} \cdot 0.3 \cdot (0.8 + 0.9) = 0.255$ and $\frac{1}{2} \cdot 0.4 \cdot (0.9 + 1) = 0.38$	The last two segments.
$\mathbf{AUC} = 0.025 + 0.13 + 0.255 + 0.38 = 0.79$	With straight lines between the points the trapezoid rule is exact, not an estimate. A random classifier's diagonal has area 0.5 and a perfect one 1.

Where this goes wrong

1. Using every point in a left sum

The table of values has $n + 1$ entries, and all of them go into the sum.

$\Delta x = \frac{1}{4}$ and heights $1, \frac{4}{5}, \frac{2}{3}, \frac{4}{7}, \frac{1}{2}$ for $\frac{1}{x}$ on $[1, 2]$	Right so far: Problem 4, step 1.
“Add the heights and multiply by Δx.”	The habit that causes the mistake: there are five points but only four strips.
$L_4 = \frac{1}{4}\left(1 + \frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{1}{2}\right) \approx 0.8845$ ✗	That counts a fifth rectangle, over $[2, \frac{9}{4}]$, outside the interval. A left sum uses the first n heights, giving 0.7595; a right sum uses the last n. Only the trapezoid rule uses all $n + 1$, and with the ends halved.

2. Writing x_i = i/n on the wrong interval

$x_i = \frac{i}{n}$ is the most familiar form, and it gets used for every interval.

$\int_0^3 (2x+1)\,dx$ with $\Delta x = \frac{3}{n}$	Right so far: Problem 5, step 1.
"$x_i = \frac{i}{n}$."	The habit that causes the mistake: the formula for $[0, 1]$ used on $[0, 3]$.
$R_n = \sum_{i=1}^{n} \left(\frac{2i}{n} + 1\right)\frac{3}{n} = 6 + \frac{3}{n} \to 6$ ✗	The points $\frac{i}{n}$ only reach 1, so this is $3\int_0^1 (2x+1)\,dx$, not the integral over $[0, 3]$. With $x_i = a + i\,\Delta x = \frac{3i}{n}$ the limit is 12.

3. Giving the endpoints full weight in the trapezoid rule

The trapezoid rule's pattern of $1, 2, 2, \ldots, 2, 1$ gets flattened to all ones, or the factor of $\frac{1}{2}$ gets lost.

Speeds $0, 8, 14, 18, 20$ every 10 s	Right so far: Problem 8.
"The trapezoid rule adds the readings and multiplies by Δt."	The habit that causes the mistake: every reading counted once, at full width.
$T_4 = 10(0 + 8 + 14 + 18 + 20) = 600$ m ✗	That is five full rectangles over four intervals. Each end reading belongs to one trapezoid and each interior reading to two, each with half weight: $5(0 + 16 + 28 + 36 + 20) = 500$ m.

4. Assuming equal widths for unequal steps

Four trapezoids on $[0, 1]$ suggests a width of $\frac{1}{4}$ each.

Points $(0, 0)$, $(0.1, 0.5)$, $(0.3, 0.8)$, $(0.6, 0.9)$, $(1, 1)$	Right so far: Problem 10.
"Four strips on $[0, 1]$, so $\Delta x = 0.25$."	The habit that causes the mistake: the equal-width formula used on data that were not sampled evenly.

$\text{AUC} = 0.25\big(\frac{0+0.5}{2} + \frac{0.5+0.8}{2} + \frac{0.8+0.9}{2} + \frac{0.9+1}{2}\big) = 0.25 \cdot 2.7 = 0.675$ ✗	The steep first segment is only 0.1 wide and the flat last one is 0.4 wide, and equal widths get both wrong. Each trapezoid needs its own width: the area is 0.79.

5. Assuming a left sum always underestimates

In the usual picture, of an increasing curve, the left rectangles sit under it.

$L_4 \approx 0.7595$ for $\int_1^2 \frac{dx}{x}$	Right so far: Problem 4, step 2.
“Left sums are too small.”	The habit that causes the mistake: a rule for increasing functions applied to every function.
So $\ln 2 > 0.7595$ ✗	$\frac{1}{x}$ is decreasing, so each left edge is the strip's highest point and L_4 is too big: $\ln 2 \approx 0.6931$. Decide over or under from whether f increases or decreases on the interval, and if it does both, the sum alone does not tell you.

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