Related rates: differentiate the relation, then substitute

Ten related-rates problems solved by differentiating an equation in time: a growing circle and sphere, a sliding ladder, a filling cone and trough, two cars, a shadow, a rising balloon's angle, the length of a moving vector, and the distance between clock hands, with worked solutions and the mistakes of substituting too early and dropping the chain rule.

BEFORE YOU START

A related-rates problem is the chain rule with time as the input. Several quantities change together, an equation ties them, and differentiating that equation with respect to t ties their rates. The same move gives the rate a loss changes along a training trajectory, $\frac{d}{dt}L(w(t)) = \nabla L \cdot \frac{dw}{dt}$. The work goes wrong in a few places: a value substituted before differentiating, which freezes a quantity that is changing, a missing $\frac{dx}{dt}$ from the chain rule, a rate given the wrong sign, and an answer to a different question from the one asked. These ten problems cover each of them.

- Every changing quantity is a function of time t. Write x' for $\frac{dx}{dt}$. A rate is positive when the quantity is increasing and negative when it is decreasing.
- The method: (1) name the quantities and write an equation that relates them and holds at **every** time; (2) differentiate both sides with respect to t, using the chain rule on each changing quantity; (3) only then substitute the values at the instant asked about, and solve for the unknown rate.
- Chain rule in time: $\frac{d}{dt}\,x^2 = 2x\,x'$, $\frac{d}{dt}\,\tan\theta = \sec^2\theta\,\theta'$, and for a product $\frac{d}{dt}(xy) = x'y + xy'$. A constant differentiates to 0.
- Formulas used: circle area $A = \pi r^2$; sphere volume $V = \frac{4}{3}\pi r^3$; cone volume $V = \frac{1}{3}\pi r^2 h$; Pythagoras; and the law of cosines $c^2 = a^2 + b^2 - 2ab\cos C$.
- Angles are in radians, so an angle's rate is in radians per unit time.

Builds on: Differentiation rules: chain, product, quotient

Problems

1. The radius of a circle grows at 2 cm/s. How fast is its area growing when the radius is 5 cm? `difficulty ·`

2. A 5 m ladder leans against a wall. Its foot slides away from the wall at 0.3 m/s. How fast is the top sliding down the wall when the foot is 3 m from the wall? difficulty ·

3. Air is pumped into a spherical balloon at 100 cm³/s. How fast is the radius increasing when the radius is 5 cm? difficulty ··

4. Water flows at 2 m³/min into a tank shaped like an inverted cone, 6 m tall with a top radius of 3 m. How fast is the water level rising when the water is 4 m deep? difficulty ··

5. Car A is 3 km north of an intersection, driving south toward it at 60 km/h. Car B is 4 km east of the intersection, driving east at 80 km/h. Is the distance between them increasing or decreasing, and how fast? difficulty ··

6. A person 1.8 m tall walks away from a 6 m lamppost at 1.4 m/s. How fast is their shadow lengthening, and how fast is the tip of the shadow moving? difficulty ··

7. A balloon is released 40 m from an observer and rises straight up at 3 m/s. How fast is the angle of elevation from the observer to the balloon increasing when the balloon is 30 m up? difficulty ···

8. A parameter vector moves with $w(t) \in \mathbb{R}^2$. At one instant $w = (3, 4)$ and $w' = (1, -2)$. How fast is its length $\|w\|$ changing? difficulty ··

9. A trough is 4 m long, and its cross-section is an inverted isosceles triangle 1 m across the top and 0.5 m deep. Water flows in at 0.1 m³/min. How fast is the level rising when the water is 0.25 m deep? difficulty ··

10. A clock's hour hand is 4 cm long and its minute hand 6 cm. How fast is the distance between their tips changing at 3:00? difficulty ···

Answers

1. $20\pi \approx 62.83$ cm²/s
2. The top slides down at 0.225 m/s
3. $r' = \dfrac{1}{\pi} \approx 0.318$ cm/s
4. $h' = \dfrac{1}{2\pi} \approx 0.159$ m/min
5. Increasing, at 28 km/h
6. The shadow lengthens at 0.6 m/s; the tip moves at 2.0 m/s
7. 0.048 rad/s
8. $\|w\|' = -1$: the length is shrinking at 1 unit per unit time
9. $h' = 0.05$ m/min
10. Decreasing, at $\dfrac{11\pi}{30\sqrt{13}} \approx 0.319$ cm/min

Worked solutions

Problem 1

> The radius of a circle grows at 2 cm/s. How fast is its area growing when the radius is 5 cm?

$A = \pi r^2$	Holds at every time.
$A' = 2\pi r\, r'$	Differentiate with respect to t; the chain rule gives the factor r'.
$A' = 2\pi \cdot 5 \cdot 2$	Now substitute $r = 5$ and $r' = 2$.
$20\pi \approx 62.83$ **cm²/s**	The area grows at the circumference times the speed of the edge.

Problem 2

> A 5 m ladder leans against a wall. Its foot slides away from the wall at 0.3 m/s. How fast is the top sliding down the wall when the foot is 3 m from the wall?

$x^2 + y^2 = 25$, with x the distance of the foot from the wall and y the height of the top.	Pythagoras; the ladder's length is constant.
$2x\,x' + 2y\,y' = 0$	Differentiate in t; the constant 25 gives 0.
At the instant: $x = 3$, $y = \sqrt{25 - 9} = 4$, $x' = 0.3$.	x' is positive because x is increasing.
$y' = -\dfrac{x\,x'}{y} = -\dfrac{3 \cdot 0.3}{4} = -0.225$	Solve step 2 for y'.
The top slides down at 0.225 **m/s**	$y' < 0$: the height is decreasing.

Problem 3

> Air is pumped into a spherical balloon at 100 cm³/s. How fast is the radius increasing when the radius is 5 cm?

$V = \frac{4}{3}\pi r^3$	Sphere volume, at every time.
$V' = 4\pi r^2\, r'$	Chain rule. $4\pi r^2$ is the surface area: the volume grows at the surface area times the speed of the surface.
$100 = 4\pi \cdot 25 \cdot r'$	Substitute $V' = 100$ and $r = 5$.
$r' = \dfrac{1}{\pi} \approx 0.318$ **cm/s**	$\frac{100}{100\pi}$. The same inflow makes the radius grow more slowly as the balloon gets bigger.

Problem 4

> Water flows at 2 m³/min into a tank shaped like an inverted cone, 6 m tall with a top radius of 3 m. How fast is the water level rising when the water is 4 m deep?

$r = \frac{h}{2}$ at every depth h	Similar triangles: the water forms a smaller cone of the same shape, with $\frac{r}{h} = \frac{3}{6}$.
$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \dfrac{\pi h^3}{12}$	Use step 1 to remove r before differentiating, so that only one changing quantity is left.
$V' = \dfrac{\pi h^2}{4}\, h'$	Chain rule.
$2 = \dfrac{\pi \cdot 16}{4}\, h'$	Substitute $V' = 2$ and $h = 4$, now and not before.
$h' = \dfrac{1}{2\pi} \approx 0.159$ **m/min**	$\frac{2}{4\pi}$. Substituting $r = 2$ before differentiating gives three times this (mistake 1).

Problem 5

> Car A is 3 km north of an intersection, driving south toward it at 60 km/h. Car B is 4 km east of the intersection, driving east at 80 km/h. Is the distance between them increasing or decreasing, and how fast?

$s^2 = x^2 + y^2$, with x B's distance east and y A's distance north of the intersection	The roads are perpendicular.
$2s\,s' = 2x\,x' + 2y\,y'$, so $s' = \dfrac{x\,x' + y\,y'}{s}$	Differentiate in t.
At the instant: $x = 4$, $x' = 80$, $y = 3$, $y' = -60$, $s = 5$.	A is driving toward the intersection, so y is decreasing and y' is negative.
$s' = \dfrac{4 \cdot 80 + 3 \cdot (-60)}{5} = \dfrac{320 - 180}{5}$	Substitute.
Increasing, at 28 km/h	$\frac{140}{5}$. B moving away outweighs A closing in.

Problem 6

> A person 1.8 m tall walks away from a 6 m lamppost at 1.4 m/s. How fast is their shadow lengthening, and how fast is the tip of the shadow moving?

$\dfrac{6}{x + s} = \dfrac{1.8}{s}$, with x the person's distance from the post and s the shadow's length	Similar triangles: the lamp and the person cast the same ray to the tip, which is $x + s$ from the post.
$6s = 1.8x + 1.8s$, so $4.2s = 1.8x$ and $s = \frac{3}{7}x$	Cross-multiply.
$s' = \frac{3}{7}x' = \frac{3}{7} \cdot 1.4 = 0.6$	Differentiate; here the rate does not depend on where the person is.
Tip: $\frac{d}{dt}(x + s) = 1.4 + 0.6 = 2.0$	The tip is at distance $x + s$ from the post.

The shadow lengthens at 0.6 m/s; the tip moves at 2.0 m/s	Two different questions, two different rates (mistake 4).

Problem 7

> A balloon is released 40 m from an observer and rises straight up at 3 m/s. How fast is the angle of elevation from the observer to the balloon increasing when the balloon is 30 m up?

$\tan\theta = \dfrac{h}{40}$	Opposite over adjacent; the 40 m is constant.
$\sec^2\theta\,\theta' = \dfrac{h'}{40}$	Chain rule on the left: $\frac{d}{dt}\tan\theta = \sec^2\theta\,\theta'$.
At $h = 30$: the line of sight is $\sqrt{30^2 + 40^2} = 50$ m, so $\cos\theta = \frac{40}{50} = 0.8$ and $\sec^2\theta = \frac{1}{0.64} = 1.5625$.	Read the cosine from the triangle instead of finding θ.
$\theta' = \dfrac{3/40}{1.5625} = \dfrac{0.075}{1.5625}$	Substitute $h' = 3$ and solve.
0.048 **rad/s**	About 2.75° per second.

Problem 8

> A parameter vector moves with $w(t) \in \mathbb{R}^2$. At one instant $w = (3, 4)$ and $w' = (1, -2)$. How fast is its length $\|w\|$ changing?

$\|w\|^2 = w \cdot w$	At every time. Work with the square to avoid differentiating a square root.
$2\|w\|\,\|w\|' = 2\,w \cdot w'$, so $\|w\|' = \dfrac{w \cdot w'}{\|w\|}$	The product rule for a dot product: $\frac{d}{dt}(w \cdot w) = w' \cdot w + w \cdot w'$.
$w \cdot w' = 3 \cdot 1 + 4 \cdot (-2) = -5$ and $\|w\| = 5$	Substitute.

$\lVert w \rVert' = -1$: **the length is shrinking at 1 unit per unit time**	Only the component of w' along w changes the length. Problem 5 is the same formula with w the cars' position difference.

Problem 9

> A trough is 4 m long, and its cross-section is an inverted isosceles triangle 1 m across the top and 0.5 m deep. Water flows in at 0.1 m³/min. How fast is the level rising when the water is 0.25 m deep?

At depth h the water's surface is $2h$ wide.	Similar triangles: the cross-section is twice as wide as it is deep, $\frac{1}{0.5} = 2$.
$V = 4 \cdot \frac{1}{2} \cdot 2h \cdot h = 4h^2$	Length times the area of the triangle of water.
$V' = 8h\,h'$	Differentiate in t.
$0.1 = 8 \cdot 0.25 \cdot h'$	Substitute $V' = 0.1$ and $h = 0.25$.
$h' = 0.05$ **m/min**	$\frac{0.1}{2}$.

Problem 10

> A clock's hour hand is 4 cm long and its minute hand 6 cm. How fast is the distance between their tips changing at 3:00?

$d^2 = 4^2 + 6^2 - 2 \cdot 4 \cdot 6 \cos\theta = 52 - 48\cos\theta$, with θ the angle between the hands	Law of cosines in the triangle formed by the hands and the segment between their tips.
$2d\,d' = 48 \sin\theta\,\theta'$	Differentiate in t; $\frac{d}{dt}\cos\theta = -\sin\theta\,\theta'$.

$\theta' = \dfrac{2\pi}{720} - \dfrac{2\pi}{60} = -\dfrac{11\pi}{360}$ rad/min	Just after 3:00 the hour hand is θ ahead of the minute hand. The minute hand turns 2π in 60 min and the hour hand in 720 min, so the gap closes at the difference of their rates.
At 3:00: $\theta = \frac{\pi}{2}$, $\sin\theta = 1$, $d = \sqrt{52} = 2\sqrt{13}$.	$\cos\frac{\pi}{2} = 0$.
$d' = \dfrac{48 \cdot 1 \cdot \left(-\frac{11\pi}{360}\right)}{2 \cdot 2\sqrt{13}} = -\dfrac{11\pi}{30\sqrt{13}}$	Solve step 2 for d'.
Decreasing, at $\dfrac{11\pi}{30\sqrt{13}} \approx 0.319$ cm/min	The minute hand is catching up with the hour hand, so the tips are getting closer.

Where this goes wrong

1. Substituting before differentiating

The numbers are given, and putting them in early makes the equation simpler.

Cone tank, $h = 4$, so the water's surface radius is $r = 2$	Right so far: Problem 4, step 1 at the instant.
"$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \cdot 4 \cdot h$, then differentiate."	The habit that causes the mistake: substituting values first, as in a problem with no rates.
$V' = \frac{4}{3}\pi\, h'$, so $h' = \dfrac{2 \cdot 3}{4\pi} = \dfrac{3}{2\pi} \approx 0.477$ m/min ✗	$r = 2$ only at this instant; r is changing with h, and fixing it freezes the cone's widening. Differentiate the relation that holds at every time, $V = \frac{\pi h^3}{12}$, first: $h' = \frac{1}{2\pi} \approx 0.159$ m/min, a third of this.

2. Dropping the chain rule's factor

Differentiating πr^2 gives $2\pi r$ in every exercise where the variable is r.

$A = \pi r^2$, $r = 5$, $r' = 2$	Right so far: Problem 1.
"The derivative of πr^2 is $2\pi r$."	The habit that causes the mistake: differentiating with respect to r when the question is about time.
$A' = 2\pi r = 10\pi \approx 31.4$ cm²/s ✗	That is $\frac{dA}{dr}$, the rate per centimeter of radius, not per second. The chain rule multiplies it by $\frac{dr}{dt} = 2$: $A' = 20\pi$ cm²/s. A rate that does not depend on how fast the radius grows cannot be right.

3. The wrong sign on an approaching rate

Speeds are given as positive numbers, so every rate gets entered as positive.

$s' = \dfrac{x\,x' + y\,y'}{s}$, with $x = 4$, $y = 3$, $s = 5$	Right so far: Problem 5, step 2.
"Car A drives at 60 km/h, so $y' = 60$."	The habit that causes the mistake: treating a speed as the rate of the variable.
$s' = \dfrac{4 \cdot 80 + 3 \cdot 60}{5} = 100$ km/h ✗	A drives toward the intersection, so its distance y is decreasing and $y' = -60$. The correct rate is $\frac{320-180}{5} = 28$ km/h. Before substituting, ask of each variable whether it is growing or shrinking.

4. Answering a different question

Problems with two moving things often ask about the one that is not the obvious unknown.

$s = \frac{3}{7}x$ and $s' = 0.6$ m/s	Right so far: Problem 6, step 3.
"The shadow's rate is found, so that is how fast the tip moves."	The habit that causes the mistake: stopping at the first rate computed.

The tip of the shadow moves at 0.6 m/s ✗	0.6 m/s is how fast the shadow gets longer, measured from the person's feet, which are themselves moving at 1.4 m/s. The tip is $x + s$ from the post, so it moves at $1.4 + 0.6 = 2.0$ m/s. Write down which distance the question is about before differentiating.

5. Differentiating with respect to x instead of t

Implicit differentiation is usually practiced with $\frac{dy}{dx}$, and the habit carries over.

$x^2 + y^2 = 25$, $x = 3$, $y = 4$	Right so far: Problem 2, step 1.
"Differentiate: $2x + 2y\,\frac{dy}{dx} = 0$."	The habit that causes the mistake: x treated as the independent variable when both x and y depend on time.
$\dfrac{dy}{dx} = -\dfrac{3}{4}$, so the top slides down at 0.75 m/s ✗	$-\frac{3}{4}$ is the change in height per meter the foot moves, not per second, and it never used the foot's speed. The chain rule in t gives $y' = \frac{dy}{dx}\,x' = -\frac{3}{4} \cdot 0.3 = -0.225$ m/s.

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