Quadratic forms and positive definite matrices

Ten problems on quadratic forms and positive definite matrices: writing a form as $x^T Ax$ with a symmetric matrix, classifying definite, semidefinite and indefinite matrices by their eigenvalues, the leading principal minors of a 3×3 matrix, completing the square as LDL^T, a Cholesky factor by hand, the range of a parameter that keeps a matrix positive definite, the largest and smallest values on the unit circle, the axes of an ellipse, why A^TA is positive semidefinite and when it is definite, and the Hessian of a ridge regression loss, with worked solutions and the mistakes of the unhalved cross term, a positive diagonal taken as proof, leading minors for a semidefinite test, the long axis along the largest eigenvalue and the extremes read off the diagonal.

BEFORE YOU START

A quadratic form is a sum of squares and cross products, such as $3x^2 + 4xy + 2y^2$, and every one of them can be written as $x^\top Ax$ with a symmetric matrix A. The question that matters is its sign: a form that is positive for every nonzero x is positive definite, and that single property is what makes a critical point a minimum, a loss convex with a unique minimizer, a covariance matrix valid and a Cholesky factor exist. These ten problems build the matrix of a form, test definiteness three ways (eigenvalues, leading principal minors and completing the square), factor a matrix by Cholesky, find the extremes of a form on the unit circle and the axes of an ellipse, and end with why ridge regression always has exactly one solution.

- A **quadratic form** is $q(x) = x^\top Ax = \sum_{i,j} A_{ij} x_i x_j$ with A symmetric. The coefficient of x_i^2 is A_{ii}, and the coefficient of $x_i x_j$ for $i \neq j$ is $2A_{ij}$, so the off-diagonal entries are half the cross coefficients.
- A is **positive definite** (PD) if $x^\top Ax > 0$ for every $x \neq 0$, **positive semidefinite** (PSD) if $x^\top Ax \geq 0$ for every x, negative (semi)definite if $-A$ is, and **indefinite** if $x^\top Ax$ takes both signs.
- **Eigenvalue test.** A symmetric A is $Q\Lambda Q^\top$ with Q orthogonal, so with $y = Q^\top x$, $x^\top Ax = \sum_i \lambda_i y_i^2$. A is PD exactly when every $\lambda_i > 0$, PSD when every $\lambda_i \geq 0$, and indefinite when there are eigenvalues of both signs.
- **Sylvester's criterion.** A is PD exactly when its leading principal minors (the determinants of the top-left 1×1, 2×2, ..., $n \times n$ blocks) are all positive. For semidefiniteness the leading minors are not enough: *every* principal minor must be ≥ 0.
- **Cholesky.** A is PD exactly when $A = LL^\top$ with L lower-triangular and its diagonal positive. Equivalently $A = LDL^\top$ with L unit lower-triangular and the pivots of elimination on the diagonal of D, all positive.

- **Rayleigh quotient.** For symmetric A, the largest and smallest values of $x^\top Ax$ over unit vectors x are $\lambda_{\max}$ and $\lambda_{\min}$, reached at their eigenvectors. For PD A, the set $x^\top Ax = c$ is an ellipse (ellipsoid) with axes along the eigenvectors and semi-axis lengths $\sqrt{c/\lambda_i}$.
- For any matrix M, $v^\top M^\top Mv = \|Mv\|^2$.

Builds on: Eigenvalues and eigenvectors by hand, Gaussian elimination, inverses and determinants by hand

Problems

1. Write $q(x,y) = 3x^2 + 4xy + 2y^2$ as $x^\top Ax$ with A symmetric, and check it at $(1,-1)$. Then write the matrix of $x^2 + 2y^2 + 3z^2 + 2xy - 4yz$. difficulty ·

2. Classify each matrix as positive definite, positive semidefinite or indefinite from its eigenvalues: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. difficulty ·

3. Use the leading principal minors to show that $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ is positive definite, and confirm with its eigenvalues. difficulty ··

4. Complete the square in $q(x,y) = 3x^2 + 4xy + 2y^2$, and write the result as $A = LDL^\top$. difficulty ··

5. Find the Cholesky factor L of $A = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 10 & 2 \\ -2 & 2 & 6 \end{pmatrix}$, and use it to get $\det A$. difficulty ··

6. For which real k is $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & k \\ 0 & k & 2 \end{pmatrix}$ positive definite? difficulty ··

7. Find the largest and smallest values of $q(x,y) = 5x^2 + 4xy + 2y^2$ on the unit circle $x^2 + y^2 = 1$, and where they occur. difficulty ··

8. Find the axes, the semi-axis lengths and the area of the ellipse $5x^2 - 4xy + 5y^2 = 21$. difficulty ··

9. Show that $M^\top M$ is positive semidefinite for any matrix M, and positive definite exactly when the columns of M are independent. Test $M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}$. difficulty ···

10. Ridge regression minimizes $L(w) = \|Xw - y\|^2 + \lambda \|w\|^2$. Find the gradient and Hessian of L. With $X = M_2$ from Problem 9, $y = (1, 2, 3)$ and $\lambda = \tfrac{1}{2}$, show that the Hessian is positive definite and find the unique minimizer. difficulty ···

Answers

1. $A = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$; $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$
2. A is positive definite, B is indefinite, C is positive semidefinite (not definite)
3. $\Delta_1, \Delta_2, \Delta_3 = 2, 3, 4$; the eigenvalues are $2 - \sqrt{2} \approx 0.586$, 2 and $2 + \sqrt{2} \approx 3.414$, all positive
4. $q = 3\big(x + \tfrac{2}{3}y\big)^2 + \tfrac{2}{3}y^2$, $A = LDL^\top$ with $L = \begin{pmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{pmatrix}$, $D = \operatorname{diag}(3, \tfrac{2}{3})$: positive definite
5. $L = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 1 & 2 \end{pmatrix}$, $\det A = (2 \cdot 3 \cdot 2)^2 = 144$
6. $-\sqrt{3} < k < \sqrt{3}$
7. Maximum 6 at $\pm\frac{1}{\sqrt{5}}(2, 1)$; minimum 1 at $\pm\frac{1}{\sqrt{5}}(1, -2)$
8. Major semi-axis $\sqrt{7}$ along $(1, 1)$, minor semi-axis $\sqrt{3}$ along $(1, -1)$; area $\pi\sqrt{7}\sqrt{3} = \pi\sqrt{21} \approx 14.40$
9. $M^\top M$ is always PSD; $M_1^\top M_1$ is PD, $M_2^\top M_2$ is only PSD, with $v^\top M_2^\top M_2 v = 0$ at $v = (2, -1)$
10. $\nabla^2 L = 2(X^\top X + \lambda I)$, PD with eigenvalues 1 and 31; $w^* = \big(\frac{12}{31}, \frac{24}{31}\big)$

Worked solutions

Problem 1

> Write $q(x, y) = 3x^2 + 4xy + 2y^2$ as $x^\top A x$ with A symmetric, and check it at $(1, -1)$. Then write the matrix of $x^2 + 2y^2 + 3z^2 + 2xy - 4yz$.

$A_{11} = 3$ and $A_{22} = 2$	The squares go on the diagonal.
$2A_{12} = 4$, so $A_{12} = A_{21} = 2$	The cross term $4xy$ is $A_{12}xy + A_{21}yx$, split evenly between the two off-diagonal entries.
$A = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$	Symmetric, as a form's matrix should be.
At $(1, -1)$: $q = 3 - 4 + 2 = 1$, and $x^\top A x = (1, -1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$	$Ax = (3 - 2, 2 - 2) = (1, 0)$. Using 4 off the diagonal instead gives -3 (Mistake 1).
$A = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$; $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$	In the second form, $2xy$ gives $A_{12} = 1$, $-4yz$ gives $A_{23} = -2$, and there is no xz term, so $A_{13} = 0$.

Problem 2

> Classify each matrix as positive definite, positive semidefinite or indefinite from its eigenvalues: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

A: $\lambda^2 - 4\lambda + 3 = 0$, so $\lambda = 1, 3$	Trace 4, determinant 3.
B: $\lambda^2 - 2\lambda - 3 = 0$, so $\lambda = 3, -1$	Trace 2, determinant -3. A negative determinant of a 2×2 always means one eigenvalue of each sign.
C: $\lambda^2 - 5\lambda = 0$, so $\lambda = 0, 5$	Determinant 0: the rows are proportional.

$q_B(1,-1) = 1 - 4 + 1 = -2$ and $q_C(2,-1) = 4 - 8 + 4 = 0$	A witness for each: $(1,-1)$ is B's eigenvector for -1, and $(2,-1)$ spans C's null space.
A is positive definite, B is indefinite, C is positive semidefinite (not definite)	B has positive diagonal entries and is still not PD (Mistake 2).

Problem 3

Use the leading principal minors to show that $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ is positive definite, and confirm with its eigenvalues.

$\Delta_1 = 2$	The top-left entry.
$\Delta_2 = 2 \cdot 2 - (-1)(-1) = 3$	The top-left 2×2 block.
$\Delta_3 = 2(4-1) - (-1)(-2-0) + 0 = 6 - 2 = 4$	Expand along the first row.
All three are positive, so A is PD by Sylvester's criterion	This is the second-difference matrix behind a discretized $-u''$.
$\Delta_1, \Delta_2, \Delta_3 = 2, 3, 4$; **the eigenvalues are** $2 - \sqrt{2} \approx 0.586$, 2 **and** $2 + \sqrt{2} \approx 3.414$, **all positive**	Their product is $4 = \det A$ and their sum is $6 = \operatorname{tr} A$.

Problem 4

Complete the square in $q(x,y) = 3x^2 + 4xy + 2y^2$, and write the result as $A = LDL^\top$.

$3x^2 + 4xy = 3\big(x^2 + \tfrac{4}{3}xy\big) = 3\big(x + \tfrac{2}{3}y\big)^2 - \tfrac{4}{3}y^2$	Factor out the x^2 coefficient, then half of $\tfrac{4}{3}$ is $\tfrac{2}{3}$, and $3 \cdot \tfrac{4}{9} = \tfrac{4}{3}$.

$q = 3\left(x + \frac{2}{3}y\right)^2 + \frac{2}{3}y^2$	$2 - \frac{4}{3} = \frac{2}{3}$.
Both coefficients are positive, so $q > 0$ unless $y = 0$ and $x + \frac{2}{3}y = 0$, that is unless $x = y = 0$	A sum of positive multiples of squares of independent combinations.
$L = \begin{pmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{pmatrix}$, $D = \operatorname{diag}\left(3, \frac{2}{3}\right)$ $\underset{L}{\begin{bmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{bmatrix}} \cdot \underset{D}{\begin{bmatrix} 3 & 0 \\ 0 & \frac{2}{3} \end{bmatrix}} \cdot \underset{L^\top}{\begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 1 \end{bmatrix}} = \underset{A}{\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}}$	The rows of $L^\top$ are the combinations being squared; the entries of D are the pivots of elimination on A.
$q = 3\left(x + \frac{2}{3}y\right)^2 + \frac{2}{3}y^2$, $A = LDL^\top$ **with** $L = \begin{pmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{pmatrix}$, $D = \operatorname{diag}(3, \frac{2}{3})$: **positive definite**	Check: $\det A = 3 \cdot \frac{2}{3} = 2 = 6 - 4$.

Problem 5

Find the Cholesky factor L of $A = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 10 & 2 \\ -2 & 2 & 6 \end{pmatrix}$, and use it to get $\det A$.

$l_{11} = \sqrt{4} = 2,\ l_{21} = \frac{2}{2} = 1,\ l_{31} = \frac{-2}{2} = -1$	The first column of A is l_{11} times the first column of L.
$l_{22} = \sqrt{10 - l_{21}^2} = \sqrt{9} = 3$	Subtract what the first column already supplies, then take the root.
$l_{32} = \dfrac{2 - l_{31}l_{21}}{l_{22}} = \dfrac{2 - (-1)(1)}{3} = 1$	From $A_{32} = l_{31}l_{21} + l_{32}l_{22}$.

$l_{33} = \sqrt{6 - l_{31}^2 - l_{32}^2} = \sqrt{6 - 1 - 1} = 2$	Every square root was of a positive number, which is the proof that A is PD.
$\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 1 & 2 \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}}_{L^\top} = \underbrace{\begin{bmatrix} 4 & 2 & -2 \\ 2 & 10 & 2 \\ -2 & 2 & 6 \end{bmatrix}}_{A}$	
$L = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 1 & 2 \end{pmatrix}$, $\det A = (2 \cdot 3 \cdot 2)^2 =$ 144	$\det A = \det L \, \det L^\top$. Sampling $\mathcal{N}(0, A)$ is $L\varepsilon$ with $\varepsilon \sim \mathcal{N}(0, I)$.

Problem 6

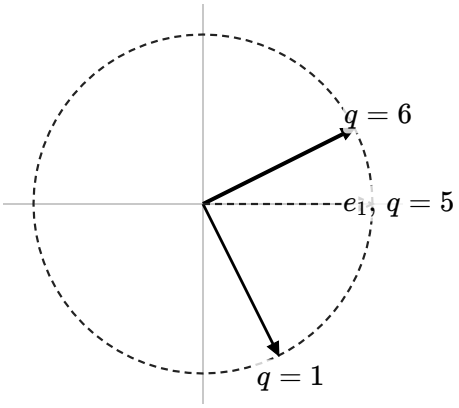
> For which real k is $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & k \\ 0 & k & 2 \end{pmatrix}$ positive definite?

$\Delta_1 = 2 > 0$ and $\Delta_2 = 4 - 1 = 3 > 0$	These do not involve k.
$\Delta_3 = 2(4 - k^2) - 1(2 - 0) + 0 = 6 - 2k^2$	Expand along the first row.
$6 - 2k^2 > 0 \iff k^2 < 3$	Sylvester's criterion needs all three minors positive.
At $k = \sqrt{3}$ the determinant is 0 and the smallest eigenvalue is 0: PSD but not PD	The boundary of the range.
$-\sqrt{3} < k < \sqrt{3}$	Check $k = 2$: $(1, -2, 2)$ gives $x^\top A x = 2 - 4 + 8 + 8 - 16 = -2 < 0$.

Problem 7

> Find the largest and smallest values of $q(x, y) = 5x^2 + 4xy + 2y^2$ on the unit circle $x^2 + y^2 = 1$, and where they occur.

$A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$	Half of 4 off the diagonal.
$\lambda^2 - 7\lambda + 6 = 0$, so $\lambda = 6, 1$	Trace 7, determinant $10 - 4 = 6$.
$\lambda = 6$: $(A - 6I)v = 0$ with rows $(-1, 2)$, so $v = \frac{1}{\sqrt{5}}(2, 1)$	Check: $q = \frac{1}{5}(20 + 8 + 2) = 6$.
$\lambda = 1$: rows $(4, 2)$, so $v = \frac{1}{\sqrt{5}}(1, -2)$	Check: $q = \frac{1}{5}(5 - 8 + 8) = 1$.

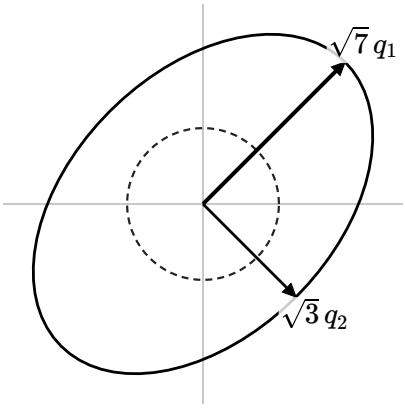


Maximum 6 **at** $\pm\frac{1}{\sqrt{5}}(2, 1)$; **minimum** 1 **at** $\pm\frac{1}{\sqrt{5}}(1, -2)$	Every other unit vector gives a value in between, for example $q(1, 0) = 5$. The smallest diagonal entry, 2, is not the minimum (Mistake 5).

Problem 8

> Find the axes, the semi-axis lengths and the area of the ellipse $5x^2 - 4xy + 5y^2 = 21$.

$A = \begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}$ has $\lambda = 3$ for $(1, 1)/\sqrt{2}$ and $\lambda = 7$ for $(1, -1)/\sqrt{2}$	$A(1, 1) = (3, 3)$ and $A(1, -1) = (7, -7)$.

In the rotated coordinates $y = Q^\top x$ the ellipse is $3y_1^2 + 7y_2^2 = 21$	The cross term disappears along the eigenvectors.
Semi-axes $\sqrt{21/3} = \sqrt{7} \approx 2.646$ along $(1, 1)$ and $\sqrt{21/7} = \sqrt{3} \approx 1.732$ along $(1, -1)$ 	Divide by the eigenvalue: the *smaller* eigenvalue gives the *longer* axis.
Check the end of the long axis, $\sqrt{7} \cdot \frac{1}{\sqrt{2}}(1, 1)$, where $x = y = \sqrt{3.5}$: $(5 - 4 + 5) \cdot 3.5 = 21$	It lies on the ellipse.
Major semi-axis $\sqrt{7}$ along $(1, 1)$, **minor semi-axis $\sqrt{3}$ along** $(1, -1)$; **area** $\pi\sqrt{7}\sqrt{3} = \pi\sqrt{21} \approx 14.40$	Putting the long axis along the eigenvector of 7 gives a point where the form is 49 (Mistake 4).

Problem 9

Show that $M^\top M$ is positive semidefinite for any matrix M, and positive definite exactly when the columns of M are independent. Test $M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}$.

$v^\top M^\top M v = (Mv)^\top (Mv) = \|Mv\|^2 \geq 0$	A squared length is never negative, so $M^\top M$ is PSD.

$\|Mv\|^2 = 0 \iff Mv = 0$, and Mv is the combination of the columns with coefficients v	So $v^\top M^\top M v > 0$ for every $v \neq 0$ exactly when no nonzero combination of the columns is 0.
$M_1^\top M_1 = \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}$, with minors 3 and $42 - 36 = 6$	Both positive: PD, as the columns $(1,1,1)$ and $(1,2,3)$ are independent.
$M_2^\top M_2 = \begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix}$, with determinant 0 and eigenvalues 0 and 15	The second column is twice the first, and $v = (2,-1)$ gives $M_2 v = 0$.
$\underset{M_2}{\begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}} \cdot \underset{v}{\begin{bmatrix} 2 \\ -1 \end{bmatrix}} = \underset{M_2 v}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$	
$M^\top M$ is always PSD; $M_1^\top M_1$ is PD, $M_2^\top M_2$ is only PSD, with $v^\top M_2^\top M_2 v = 0$ at $v = (2,-1)$	This is why the normal equations have a unique solution exactly when the features are not collinear.

Problem 10

> Ridge regression minimizes $L(w) = \|Xw - y\|^2 + \lambda \|w\|^2$. Find the gradient and Hessian of L. With $X = M_2$ from Problem 9, $y = (1,2,3)$ and $\lambda = \frac{1}{2}$, show that the Hessian is positive definite and find the unique minimizer.

$\nabla L = 2X^\top (Xw - y) + 2\lambda w$ and $\nabla^2 L = 2(X^\top X + \lambda I)$	$L = w^\top (X^\top X + \lambda I) w - 2 y^\top X w + y^\top y$, a quadratic form plus a linear term.
$v^\top (X^\top X + \lambda I) v = \|Xv\|^2 + \lambda \|v\|^2 > 0$ for $v \neq 0$	Whatever X is, when $\lambda > 0$. So L is strictly convex with one minimizer.

$X^\top X + \frac{1}{2} I = \begin{pmatrix} 3.5 & 6 \\ 6 & 12.5 \end{pmatrix}$, with eigenvalues 0.5 and 15.5, and $X^\top y = (6, 12)$	The eigenvalues of $X^\top X$, 0 and 15, each move up by λ.
$\det = 43.75 - 36 = 7.75$, so $w_1 = \dfrac{12.5 \cdot 6 - 6 \cdot 12}{7.75} = \dfrac{12}{31}$ and $w_2 = \dfrac{3.5 \cdot 12 - 6 \cdot 6}{7.75} = \dfrac{24}{31}$	Cramer's rule on $(X^\top X + \lambda I)w = X^\top y$.
$\nabla^2 L = 2(X^\top X + \lambda I)$, **PD with eigenvalues** 1 **and** 31; $w^* = \left(\frac{12}{31}, \frac{24}{31}\right)$	Without λ the Hessian is singular and every w with $w_1 + 2w_2 = 2$ is a minimizer. Ridge picks one along $(1, 2)$, with $w_1 + 2w_2 = \frac{60}{31} \approx 1.935$, shrunk slightly below 2.

Where this goes wrong

1. The cross coefficient left unhalved

The coefficient of xy is copied straight into the matrix.

$q = 3x^2 + 4xy + 2y^2$, with 3 and 2 on the diagonal	Right so far: Problem 1.
“The xy coefficient goes off the diagonal: $A = \begin{pmatrix} 3 & 4 \\ 4 & 2 \end{pmatrix}$.”	Both off-diagonal entries multiply xy, so together they give $8xy$.
The form is indefinite, since $\det A = 6 - 16 = -10 < 0$ ✗	That matrix is the form $3x^2 + 8xy + 2y^2$. At $(1, -1)$ it gives -3, while $q(1, -1) = 1$. With $A_{12} = 2$ the determinant is $2 > 0$ and q is positive definite (Problem 4).

2. A positive diagonal taken as proof

A diagonal matrix with positive entries is positive definite, and the shortcut gets applied to every matrix.

$B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ has $B_{11} = B_{22} = 1 > 0$	Right so far: Problem 2. A positive diagonal is *necessary*, since $e_i^\top B e_i = B_{ii}$.
"Every diagonal entry is positive, so every $x^\top B x$ is positive."	The cross term $4xy$ can outweigh the squares.
B is positive definite ✗	$q_B(1, -1) = 1 - 4 + 1 = -2$, and the eigenvalues are 3 and -1. A positive diagonal is not sufficient; a 2×2 also needs $\det > 0$, and here $\det B = -3$.

3. Leading minors used to test semidefiniteness

Sylvester's criterion is remembered as "check the leading minors", with > 0 relaxed to ≥ 0.

$A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ has $\Delta_1 = 0$ and $\Delta_2 = 0$	Right so far: both leading minors are ≥ 0.
"All leading minors are non-negative, so A is PSD."	For semidefiniteness every principal minor must be checked, including the bottom-right entry -1.
A is positive semidefinite ✗	$(0, 1)^\top A (0, 1) = -1 < 0$: A is negative semidefinite. The 1×1 principal minor $A_{22} = -1$ is the one the leading minors skip.

4. The long axis along the largest eigenvalue

The biggest eigenvalue is thought of as the biggest direction, as it is for a covariance matrix.

$5x^2 - 4xy + 5y^2 = 21$ has eigenvalues 3 along $(1, 1)$ and 7 along $(1, -1)$	Right so far: Problem 8.
"The major semi-axis $\sqrt{7}$ points along the eigenvector of 7."	Along an eigenvector the form is λr^2, so a large λ reaches 21 at a small r.

The ellipse reaches $\sqrt{7} \cdot \frac{1}{\sqrt{2}}(1, -1)$ ✗	There $x = -y = \sqrt{3.5}$ and the form is $(5 + 4 + 5) \cdot 3.5 = 49$, not 21. The semi-axis along an eigenvector is $\sqrt{c/\lambda}$: $\sqrt{21/7} = \sqrt{3}$ along $(1, -1)$ and $\sqrt{7}$ along $(1, 1)$. For a Gaussian the ellipse is $x^\top \Sigma^{-1} x = c$, whose eigenvalues are $1/\lambda$, which is why the covariance's largest eigenvalue does give the long axis.

5. The extremes read off the diagonal

On the unit circle the values at $(1, 0)$ and $(0, 1)$ are the diagonal entries, and these are taken as the extremes.

$q = 5x^2 + 4xy + 2y^2$ has $q(1, 0) = 5$ and $q(0, 1) = 2$	Right so far: Problem 7.
"The coordinate axes give the largest and smallest values."	Only when the cross term is 0, so that the axes are the eigenvectors.
On the unit circle, $2 \le q \le 5$ ✗	The eigenvalues are 6 and 1, at $\frac{1}{\sqrt{5}}(2, 1)$ and $\frac{1}{\sqrt{5}}(1, -2)$. The diagonal entries always lie between $\lambda_{\min}$ and $\lambda_{\max}$, so they never show the extremes unless the matrix is diagonal.

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