Partial fractions and integrating rational functions

Ten rational functions split into partial fractions and integrated: distinct linear factors by the cover-up rule, three factors, a repeated factor, an irreducible quadratic, an improper fraction that needs long division first, and the definite integral of $1/(x^2-1)$, with worked solutions and the mistakes that choose the wrong form or skip the division.

BEFORE YOU START

Substitution and parts do not integrate $\frac{1}{x^2-1}$, but splitting it into $\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)$ turns it into two logs. Partial fractions is that split, done systematically: the form of the answer is fixed by the factors of the denominator, and only the constants are unknown. The work goes wrong in the set-up more than in the arithmetic: a repeated or quadratic factor given too few terms, an improper fraction decomposed without dividing first, a cover-up evaluated at the wrong root. These ten problems decompose first, then integrate.

- A rational function is $\frac{P(x)}{Q(x)}$ with P and Q polynomials. It is **proper** when $\deg P < \deg Q$. If it is not, divide first: $\frac{P}{Q} = S + \frac{R}{Q}$ with $\deg R < \deg Q$, and decompose only $\frac{R}{Q}$.
- Factor Q completely into linear factors and irreducible quadratics (quadratics with no real roots, such as x^2+1). Each factor contributes terms to the form:

Factor of Q	Terms in the decomposition
$x-a$, once	$\frac{A}{x-a}$
$(x-a)^2$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
x^2+px+q, irreducible	$\frac{Bx+C}{x^2+px+q}$

- To find the constants, multiply both sides by Q and either substitute convenient values of x (the roots of Q first) or match coefficients of each power of x. The **cover-up rule** is the substitution done in one step: for a factor $x-a$ that appears once, A is $\frac{P}{Q}$ with the factor $x-a$ covered up, evaluated at $x=a$. For the factor $x+2$, $a=-2$.
- Integrals used: $\int \frac{dx}{x-a} = \ln|x-a|+C$; $\int \frac{dx}{(x-a)^2} = -\frac{1}{x-a}+C$; $\int \frac{2x}{x^2+1}dx = \ln(x^2+1)+C$; $\int \frac{dx}{x^2+1} = \arctan x + C$.

Builds on: Integration: substitution and parts

Problems

1. Decompose $\dfrac{5x-1}{(x+1)(x-2)}$ into partial fractions. difficulty ·

2. Decompose $\dfrac{x+7}{x^2-x-6}$. difficulty ·

3. Decompose $\dfrac{2x^2+6x-2}{x(x-1)(x+2)}$. difficulty ··

4. Decompose $\dfrac{x^2+3}{x(x-1)^2}$. difficulty ··

5. Decompose $\dfrac{2x^2+3x+3}{(x+1)(x^2+1)}$. difficulty ··

6. Write $\dfrac{x^3+x^2+x+3}{x^2-1}$ as a polynomial plus partial fractions. difficulty ··

7. Find $\displaystyle\int \frac{5x-1}{(x+1)(x-2)}\,dx$. difficulty ··

8. Evaluate $\displaystyle\int_2^3 \frac{dx}{x^2-1}$ exactly. difficulty ···

9. Find $\displaystyle\int \frac{x^2+3}{x(x-1)^2}\,dx$. difficulty ···

10. Find $\displaystyle\int \frac{2x^2+3x+3}{(x+1)(x^2+1)}\,dx$. difficulty ···

Answers

1. $\dfrac{2}{x+1} + \dfrac{3}{x-2}$
2. $\dfrac{2}{x-3} - \dfrac{1}{x+2}$
3. $\dfrac{1}{x} + \dfrac{2}{x-1} - \dfrac{1}{x+2}$
4. $\dfrac{3}{x} - \dfrac{2}{x-1} + \dfrac{4}{(x-1)^2}$
5. $\dfrac{1}{x+1} + \dfrac{x+2}{x^2+1}$
6. $x + 1 + \dfrac{3}{x-1} - \dfrac{1}{x+1}$
7. $2\ln|x+1| + 3\ln|x-2| + C$
8. $\frac{1}{2}\ln\frac{3}{2} \approx 0.2027$
9. $3\ln|x| - 2\ln|x-1| - \dfrac{4}{x-1} + C$
10. $\ln|x+1| + \frac{1}{2}\ln(x^2+1) + 2\arctan x + C$

Worked solutions

Problem 1

> Decompose $\dfrac{5x-1}{(x+1)(x-2)}$ into partial fractions.

$\dfrac{5x-1}{(x+1)(x-2)} = \dfrac{A}{x+1} + \dfrac{B}{x-2}$	Two distinct linear factors, one constant over each. The fraction is proper: degree 1 over degree 2.
$5x-1 = A(x-2) + B(x+1)$	Multiply both sides by $(x+1)(x-2)$. This holds for every x.
$x = 2$: $9 = 3B$, so $B = 3$.	At $x = 2$ the A term is 0. This is the cover-up rule: cover $x-2$ and evaluate $\frac{5x-1}{x+1}$ at 2.
$x = -1$: $-6 = -3A$, so $A = 2$.	At $x = -1$ the B term is 0.
$\dfrac{2}{x+1} + \dfrac{3}{x-2}$	Check: $2(x-2) + 3(x+1) = 5x - 1$.

Problem 2

> Decompose $\dfrac{x+7}{x^2-x-6}$.

$x^2 - x - 6 = (x-3)(x+2)$	Factor the denominator first: -3 and 2 multiply to -6 and add to -1.
$\dfrac{x+7}{(x-3)(x+2)} = \dfrac{A}{x-3} + \dfrac{B}{x+2}$	Two distinct linear factors.
$A = \dfrac{3+7}{3+2} = 2$	Cover-up for $x-3$: cover it and evaluate $\frac{x+7}{x+2}$ at its root, $x = 3$.
$B = \dfrac{-2+7}{-2-3} = \dfrac{5}{-5} = -1$	Cover-up for $x+2$: its root is $x = -2$, so evaluate $\frac{x+7}{x-3}$ there.
$\dfrac{2}{x-3} - \dfrac{1}{x+2}$	Check: $2(x+2) - (x-3) = x + 7$.

Problem 3

> Decompose $\dfrac{2x^2+6x-2}{x(x-1)(x+2)}$.

$\dfrac{2x^2+6x-2}{x(x-1)(x+2)} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x+2}$	Three distinct linear factors; degree 2 over degree 3, so proper.
$A = \dfrac{-2}{(0-1)(0+2)} = \dfrac{-2}{-2} = 1$	Cover up x and evaluate the rest at $x = 0$.
$B = \dfrac{2+6-2}{1 \cdot (1+2)} = \dfrac{6}{3} = 2$	Cover up $x-1$ and evaluate at $x = 1$.
$C = \dfrac{8-12-2}{(-2)(-2-1)} = \dfrac{-6}{6} = -1$	Cover up $x+2$ and evaluate at $x = -2$.
$\dfrac{1}{x} + \dfrac{2}{x-1} - \dfrac{1}{x+2}$	Check: $(x-1)(x+2) + 2x(x+2) - x(x-1) = (x^2+x-2) + (2x^2+4x) - (x^2-x) = 2x^2+6x-2$.

Problem 4

> Decompose $\dfrac{x^2+3}{x(x-1)^2}$.

$\dfrac{x^2+3}{x(x-1)^2} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$	The repeated factor $(x-1)^2$ needs a term for each power, $x-1$ and $(x-1)^2$. Three unknowns for a cubic denominator.
$x^2+3 = A(x-1)^2 + Bx(x-1) + Cx$	Multiply both sides by $x(x-1)^2$.
$x = 0$: $3 = A$. $x = 1$: $4 = C$.	Each root kills every term but one.
Coefficient of x^2: $1 = A + B$, so $B = -2$.	No root isolates B, so match the x^2 terms: $A(x-1)^2$ gives Ax^2, $Bx(x-1)$ gives Bx^2, and Cx has none.
$\dfrac{3}{x} - \dfrac{2}{x-1} + \dfrac{4}{(x-1)^2}$	Check at $x = 2$: the left side is $\frac{7}{2}$ and the right side is $\frac{3}{2} - 2 + 4 = \frac{7}{2}$.

Problem 5

> Decompose $\frac{2x^2+3x+3}{(x+1)(x^2+1)}$.

$\frac{2x^2+3x+3}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$	x^2+1 has no real roots, so it stays as a factor, and its numerator is a general linear polynomial $Bx+C$, not a constant.
$2x^2+3x+3 = A(x^2+1) + (Bx+C)(x+1)$	Multiply both sides by $(x+1)(x^2+1)$.
$x=-1$: $2-3+3=2A$, so $A=1$.	The root of the linear factor.
x^2: $2=A+B$, so $B=1$. Constant: $3=A+C$, so $C=2$.	Match coefficients. $(Bx+C)(x+1) = Bx^2+(B+C)x+C$.
$\frac{1}{x+1} + \frac{x+2}{x^2+1}$	Check the x coefficient, which was not used: $B+C=3$, as required.

Problem 6

> Write $\frac{x^3+x^2+x+3}{x^2-1}$ as a polynomial plus partial fractions.

The fraction is improper: degree 3 over degree 2.	Divide before decomposing.
$x^3+x^2+x+3 = (x+1)(x^2-1) + (2x+4)$	Long division: $x\cdot(x^2-1) = x^3-x$ leaves x^2+2x+3; $1\cdot(x^2-1)$ leaves $2x+4$, of degree less than 2. Quotient $x+1$, remainder $2x+4$.
$\frac{x^3+x^2+x+3}{x^2-1} = x+1+\frac{2x+4}{(x-1)(x+1)}$	Step 2 divided by x^2-1.
$\frac{2x+4}{(x-1)(x+1)} = \frac{3}{x-1} - \frac{1}{x+1}$	Cover-up: at $x=1$, $\frac{6}{2}=3$; at $x=-1$, $\frac{2}{-2}=-1$.

$x+1+\dfrac{3}{x-1}-\dfrac{1}{x+1}$	Check at $x=2$: the left side is $\frac{17}{3}$ and the right side is $3+3-\frac{1}{3}=\frac{17}{3}$.

Problem 7

> Find $\displaystyle\int \frac{5x-1}{(x+1)(x-2)}\,dx$.

$\displaystyle\int\left(\frac{2}{x+1}+\frac{3}{x-2}\right)dx$	The decomposition from Problem 1.
$\displaystyle\int\frac{2}{x+1}dx=2\ln\lvert x+1\rvert$ and $\displaystyle\int\frac{3}{x-2}dx=3\ln\lvert x-2\rvert$.	$\int\frac{dx}{x-a}=\ln\lvert x-a\rvert$, with the constant carried along. The absolute value keeps the log defined on both sides of the root.
$2\ln\lvert x+1\rvert+3\ln\lvert x-2\rvert+C$	Check by differentiating: $\frac{2}{x+1}+\frac{3}{x-2}$, which is the integrand.

Problem 8

> Evaluate $\displaystyle\int_2^3\frac{dx}{x^2-1}$ exactly.

$\dfrac{1}{(x-1)(x+1)}=\dfrac{1/2}{x-1}-\dfrac{1/2}{x+1}$	Cover-up: at $x=1$, $\frac{1}{2}$; at $x=-1$, $\frac{1}{-2}$.
$\displaystyle\int\frac{dx}{x^2-1}=\tfrac{1}{2}\big(\ln\lvert x-1\rvert-\ln\lvert x+1\rvert\big)=\tfrac{1}{2}\ln\left\lvert\frac{x-1}{x+1}\right\rvert$	Integrate each term, then combine with $\ln p-\ln q=\ln\frac{p}{q}$.
At $x=3$: $\frac{1}{2}\ln\frac{2}{4}=\frac{1}{2}\ln\frac{1}{2}$. At $x=2$: $\frac{1}{2}\ln\frac{1}{3}$.	Neither root of x^2-1 lies in $[2,3]$, so the integrand is continuous there and the antiderivative can be evaluated at the ends.
$\frac{1}{2}\ln\frac{1}{2}-\frac{1}{2}\ln\frac{1}{3}=\frac{1}{2}\ln\dfrac{1/2}{1/3}$	Upper value minus lower value, then the same log rule.

$\frac{1}{2}\ln\frac{3}{2} \approx 0.2027$	Sanity check: the integrand falls from $\frac{1}{3}$ to $\frac{1}{8}$ on an interval of length 1, so the integral lies between 0.125 and 0.333.

Problem 9

Find $\displaystyle\int \frac{x^2+3}{x(x-1)^2}\,dx$.

$\displaystyle\int \left(\frac{3}{x} - \frac{2}{x-1} + \frac{4}{(x-1)^2}\right) dx$	The decomposition from Problem 4.
$\displaystyle\int \frac{3}{x}dx = 3\ln\lvert x\rvert$ and $\displaystyle\int \frac{-2}{x-1}dx = -2\ln\lvert x-1\rvert$.	The two linear terms give logs.
$\displaystyle\int \frac{4}{(x-1)^2}dx = 4\int (x-1)^{-2}dx = -\frac{4}{x-1}$	The squared term is a power, not a log: $\int u^{-2}du = -u^{-1}$ with $u = x-1$.
$3\ln\lvert x\rvert - 2\ln\lvert x-1\rvert - \frac{4}{x-1} + C$	Check the last term: $\frac{d}{dx}\big(-4(x-1)^{-1}\big) = 4(x-1)^{-2}$.

Problem 10

Find $\displaystyle\int \frac{2x^2+3x+3}{(x+1)(x^2+1)}\,dx$.

$\displaystyle\int \left(\frac{1}{x+1} + \frac{x}{x^2+1} + \frac{2}{x^2+1}\right) dx$	The decomposition from Problem 5, with $\frac{x+2}{x^2+1}$ split into its x part and its constant part, which integrate differently.
$\displaystyle\int \frac{x}{x^2+1}dx = \tfrac{1}{2}\ln(x^2+1)$	Substitution $u = x^2+1$, $du = 2x\,dx$: the numerator is half of du. No absolute value is needed because $x^2+1 > 0$.
$\displaystyle\int \frac{2}{x^2+1}dx = 2\arctan x$	The constant part gives the arctangent.

$\ln\lvert x+1\rvert + \frac{1}{2}\ln(x^2+1) + 2\arctan x + C$	Steps 2 and 3 plus $\int \frac{dx}{x+1} = \ln\lvert x+1\rvert$.

Where this goes wrong

1. Decomposing an improper fraction without dividing

The cover-up rule produces numbers for any fraction, proper or not, so nothing seems to go wrong.

$\dfrac{x^3+x^2+x+3}{(x-1)(x+1)}$	Right so far: Problem 6 with the denominator factored.
"Two linear factors, so $\frac{A}{x-1} + \frac{B}{x+1}$; cover up to get $A = \frac{6}{2} = 3$ and $B = \frac{2}{-2} = -1$."	The habit that causes the mistake: going straight to the form without comparing degrees.
$\dfrac{x^3+x^2+x+3}{x^2-1} = \dfrac{3}{x-1} - \dfrac{1}{x+1}$ ✗	The right side goes to 0 as $x \to \infty$ while the left side grows like x, so they cannot be equal; at $x = 2$ they are $\frac{17}{3}$ and $\frac{8}{3}$. A proper-fraction form can only match a proper fraction. Divide first; the missing piece is the quotient $x+1$ (Problem 6).

2. One term for a repeated factor

$(x-1)^2$ looks like a single factor, so it gets a single fraction.

$\dfrac{x^2+3}{x(x-1)^2}$	Right so far: Problem 4.
"One fraction per factor: x and $(x-1)^2$."	The habit that causes the mistake: counting factors, not powers.
$\dfrac{x^2+3}{x(x-1)^2} = \dfrac{A}{x} + \dfrac{B}{(x-1)^2}$ ✗	Multiplying out gives $x^2+3 = A(x-1)^2 + Bx$. The roots give $A = 3$ and $B = 4$, but then the x^2 coefficients say $1 = A = 3$: no constants work. A repeated factor needs one term for each power, $\frac{B}{x-1} + \frac{C}{(x-1)^2}$ (Problem 4).

3. A constant over an irreducible quadratic

Every linear factor gets a constant on top, and that pattern gets carried to $x^2 + 1$.

$\dfrac{2x^2 + 3x + 3}{(x+1)(x^2+1)}$	Right so far: Problem 5.
"Put a constant over each factor."	The habit that causes the mistake: the numerator over a quadratic needs one degree less than the quadratic, which is degree 1, not 0.
$\dfrac{2x^2 + 3x + 3}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{C}{x^2+1}$ ✗	Matching coefficients of $2x^2 + 3x + 3 = A(x^2+1) + C(x+1)$ gives $A = 2$ from x^2, $C = 3$ from x, and then $3 = A + C = 5$ from the constants: no solution. With $\frac{Bx+C}{x^2+1}$ there are three unknowns for three equations (Problem 5).

4. Evaluating the cover-up at the wrong root

The factor is $x + 2$, and 2 is the number in it.

$\dfrac{x+7}{(x-3)(x+2)} = \dfrac{A}{x-3} + \dfrac{B}{x+2}$, with $A = 2$	Right so far: Problem 2, steps 2 and 3.
"Cover up $x + 2$ and put in $x = 2$."	The habit that causes the mistake: reading the number off the factor without its sign.
$B = \dfrac{2+7}{2-3} = -9$ ✗	The cover-up works because the covered factor is 0 at the chosen x, and $x + 2$ is 0 at $x = -2$, not 2. At $x = -2$: $B = \frac{5}{-5} = -1$. A quick check catches it: $2(x+2) - 9(x-3) = -7x + 31$, not $x + 7$.

5. Integrating 1/(x − 1)² as a log

Every other term in the decomposition integrates to a log, so this one does too.

$\displaystyle\int \left(\frac{3}{x} - \frac{2}{x-1} + \frac{4}{(x-1)^2} \right) dx$	Right so far: Problem 9, step 1.

"$\int \frac{du}{u} = \ln\lvert u\rvert$ with $u = (x-1)^2$."	The habit that causes the mistake: $\int \frac{du}{u}$ needs du on top, and $du = 2(x-1)\,dx$ is not there.
$\displaystyle\int \frac{4}{(x-1)^2}\,dx = 4\ln(x-1)^2$ ✗	Differentiating the claimed answer gives $\frac{8}{x-1}$, not $\frac{4}{(x-1)^2}$. The term is the power $4(x-1)^{-2}$, whose integral is $-\frac{4}{x-1}$ (Problem 9).

More at eulerpad.app. This is one of 82 free practice sets with worked solutions at eulerpad.app/practice, alongside guides to the topics behind them.

EulerPad, the app, reads your handwritten math on iPad, iPhone and Android and checks it line by line. It's free to start.

This set is also Chapter 15 of *Calculus I & II*, a 250-problem workbook with worked solutions: eulerpad.app/workbooks

eulerpad.app/practice/partial-fractions