Parametric and polar curves: slopes, arc length and area

Ten problems on curves given by a parameter or in polar coordinates: the slope and tangent lines of x = t², y = t³ − 3t, its second derivative, an ellipse and its area, an arc length with a perfect square under the root, the cycloid's length and area, a path that runs along a segment and back, the circle r = 2 cos θ, the cardioid's area and length, the region between two polar curves, and the slope of a polar curve, with worked solutions and the mistakes of stopping at d/dt of dy/dx, dropping the absolute value in a speed and integrating over a curve traced twice.

BEFORE YOU START

Some curves are easier to describe by where a point is at each moment, $x(t)$ and $y(t)$, than by an equation in x and y: a wheel's rim, a projectile, a planet. Others are easier to describe by distance and direction from a center, $r(\theta)$. The calculus is the same as for $y = f(x)$, with the chain rule doing the translation: slopes are ratios of rates, lengths add up $\sqrt{dx^2 + dy^2}$, and areas are sums of thin strips or thin wedges. The work goes wrong in a few places: $\frac{d}{dt}\left(\frac{dy}{dx}\right)$ taken as the second derivative, an absolute value dropped from a speed, a curve traced twice counted twice, the difference of two radii squared instead of the difference of their squares, and $\frac{dr}{d\theta}$ read as the slope. These ten problems cover each of them.

- **Parametric slope.** For $x = x(t)$, $y = y(t)$: $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$ where $\dfrac{dx}{dt} \neq 0$. The tangent is horizontal where $\dfrac{dy}{dt} = 0$ and vertical where $\dfrac{dx}{dt} = 0$ (with the other derivative nonzero).
- **Second derivative.** $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$: differentiate the slope in t, then divide by $\dfrac{dx}{dt}$ again.
- **Arc length and area.** $L = \displaystyle\int_\alpha^\beta \sqrt{x'(t)^2 + y'(t)^2}\,dt$, the integral of the speed. The area under a curve traced once from left to right is $\displaystyle\int_\alpha^\beta y(t)\,x'(t)\,dt$.
- **Polar coordinates.** $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$. A polar curve is the parametric curve $x = r(\theta)\cos\theta$, $y = r(\theta)\sin\theta$.
- **Polar area and length.** The area swept as θ goes from α to β is $\dfrac{1}{2}\displaystyle\int_\alpha^\beta r^2\,d\theta$, from thin wedges of area $\frac{1}{2}r^2\,d\theta$. The length is $\displaystyle\int_\alpha^\beta \sqrt{r^2 + r'(\theta)^2}\,d\theta$. Between two curves, use $\frac{1}{2}\int \left(r_{\text{out}}^2 - r_{\text{in}}^2\right) d\theta$.

- **Polar slope.** $\dfrac{dy}{dx} = \dfrac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$, from the product rule on $r\sin\theta$ and $r\cos\theta$.

Builds on: Differentiation rules: chain, product, quotient, Arc length and surface area of revolution

Problems

1. For $x = t^2$, $y = t^3 - 3t$, find the slope and tangent line at $t = 2$, and the points where the tangent is horizontal or vertical. difficulty ·

2. For the same curve, find $\dfrac{d^2y}{dx^2}$ and its value at $t = 2$. difficulty ··

3. Identify the curve $x = 2\cos t$, $y = 3\sin t$, find its slope at $t = \frac{\pi}{4}$, and find the area it encloses. difficulty ·

4. Find the length of $x = t^2$, $y = \frac{2}{3}t^3$ for $0 \le t \le \sqrt{3}$. difficulty ··

5. A wheel of radius 1 rolls along the x-axis, and a point on its rim traces the cycloid $x = t - \sin t$, $y = 1 - \cos t$. Find the length of one arch, $0 \le t \le 2\pi$, and the area under it. difficulty ··

6. A point moves along $x = \sin^2 t$, $y = \cos^2 t$ for $0 \le t \le \pi$. How far does it travel, and how long is the curve it traces? difficulty ··

7. Show that $r = 2\cos\theta$ is a circle, and find its area with a polar integral. difficulty ·

8. Find the area enclosed by the cardioid $r = 1 + \cos\theta$, and its length. difficulty ··

9. Find the area of the region inside $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$. difficulty ···

10. Find the slope of the cardioid $r = 1 + \cos\theta$ at $\theta = \frac{\pi}{2}$. difficulty ··

Answers

1. Slope $\frac{9}{4}$, tangent $y = \frac{9}{4}x - 7$; horizontal tangents at $(1, \pm 2)$, vertical at $(0, 0)$
2. $\dfrac{d^2y}{dx^2} = \dfrac{3(t^2+1)}{4t^3}$, which is $\dfrac{15}{32}$ at $t = 2$
3. An ellipse; slope $-\frac{3}{2}$ at $t = \frac{\pi}{4}$; area 6π
4. $\frac{2}{3}(8 - 1) = \frac{14}{3} \approx 4.667$
5. Length 8, area 3π
6. It travels $2\sqrt{2} \approx 2.828$; the segment it traces has length $\sqrt{2}$
7. π
8. Area $\frac{3\pi}{2} \approx 4.712$, length 8
9. π
10. $\dfrac{dy}{dx} = \dfrac{-1}{-1} = 1$

Worked solutions

Problem 1

> For $x = t^2$, $y = t^3 - 3t$, find the slope and tangent line at $t = 2$, and the points where the tangent is horizontal or vertical.

$\dfrac{dy}{dx} = \dfrac{3t^2 - 3}{2t}$	$\frac{dy}{dt}$ over $\frac{dx}{dt}$.
At $t = 2$: slope $\dfrac{12 - 3}{4} = \dfrac{9}{4}$, at the point $(4, 2)$	$x = 4$ and $y = 8 - 6 = 2$.
Tangent: $y - 2 = \frac{9}{4}(x - 4)$, or $y = \frac{9}{4}x - 7$	Point-slope form.
Horizontal where $3t^2 - 3 = 0$, $t = \pm 1$: the points $(1, -2)$ and $(1, 2)$. Vertical where $2t = 0$, $t = 0$: the point $(0, 0)$	At $t = 0$, $\frac{dy}{dt} = -3 \neq 0$.
Slope $\frac{9}{4}$, tangent $y = \frac{9}{4}x - 7$; horizontal tangents at $(1, \pm 2)$, vertical at $(0, 0)$	The curve also crosses itself at $(3, 0)$, where $t = \pm\sqrt{3}$, with two different slopes, $\pm\sqrt{3}$.

Problem 2

> For the same curve, find $\dfrac{d^2y}{dx^2}$ and its value at $t = 2$.

$\dfrac{dy}{dx} = \dfrac{3t^2 - 3}{2t} = \dfrac{3}{2}t - \dfrac{3}{2t}$	Split the fraction before differentiating.
$\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) = \dfrac{3}{2} + \dfrac{3}{2t^2}$	The derivative of the slope with respect to t, not yet with respect to x.
$\dfrac{d^2y}{dx^2} = \dfrac{\frac{3}{2} + \frac{3}{2t^2}}{2t} = \dfrac{3(t^2 + 1)}{4t^3}$	Divide by $\frac{dx}{dt} = 2t$: the chain rule, $\frac{d}{dx} = \frac{1}{dx/dt}\frac{d}{dt}$.

$\dfrac{d^2y}{dx^2} = \dfrac{3(t^2+1)}{4t^3}$, **which is** $\dfrac{15}{32}$ **at** $t = 2$	Positive, so the curve is concave up there. It is positive for every $t > 0$ and negative for every $t < 0$.

Problem 3

> Identify the curve $x = 2\cos t$, $y = 3\sin t$, find its slope at $t = \frac{\pi}{4}$, and find the area it encloses.

$\left(\dfrac{x}{2}\right)^2 + \left(\dfrac{y}{3}\right)^2 = \cos^2 t + \sin^2 t = 1$	Eliminate t with the Pythagorean identity: an ellipse with semi-axes 2 and 3.
$\dfrac{dy}{dx} = \dfrac{3\cos t}{-2\sin t}$, which is $-\dfrac{3}{2}$ at $t = \frac{\pi}{4}$, at the point $\left(\sqrt{2}, \frac{3\sqrt{2}}{2}\right)$	$\cos\frac{\pi}{4} = \sin\frac{\pi}{4}$.
$\text{Area} = \left\lvert \displaystyle\int_0^{2\pi} y\,x'(t)\,dt \right\rvert = \left\lvert \int_0^{2\pi} 3\sin t \cdot (-2\sin t)\,dt \right\rvert = 6\int_0^{2\pi} \sin^2 t\,dt$	The curve runs counterclockwise, from right to left along the top, so the integral comes out negative.
An ellipse; slope $-\frac{3}{2}$ **at** $t = \frac{\pi}{4}$**; area** 6π	$\int_0^{2\pi} \sin^2 t\,dt = \pi$. This is πab with $a = 2$, $b = 3$.

Problem 4

> Find the length of $x = t^2$, $y = \frac{2}{3}t^3$ for $0 \le t \le \sqrt{3}$.

$x' = 2t$, $y' = 2t^2$, so $x'^2 + y'^2 = 4t^2 + 4t^4 = 4t^2(1+t^2)$	Factor before taking the root.
$\text{Speed} = 2t\sqrt{1+t^2}$	$t \ge 0$ here, so $\sqrt{4t^2} = 2t$.
$L = \displaystyle\int_0^{\sqrt{3}} 2t\sqrt{1+t^2}\,dt = \frac{2}{3}\Big[(1+t^2)^{3/2}\Big]_0^{\sqrt{3}}$	Substitute $u = 1 + t^2$, $du = 2t\,dt$.
$\frac{2}{3}(8-1) = \frac{14}{3} \approx 4.667$	The curve runs from $(0,0)$ to $(3, 2\sqrt{3})$, and the chord between them is $\sqrt{21} \approx 4.583$.

Problem 5

A wheel of radius 1 rolls along the x-axis, and a point on its rim traces the cycloid $x = t - \sin t$, $y = 1 - \cos t$. Find the length of one arch, $0 \le t \le 2\pi$, and the area under it.

$x'^2 + y'^2 = (1 - \cos t)^2 + \sin^2 t = 2 - 2\cos t = 4\sin^2 \frac{t}{2}$	The half-angle identity $1 - \cos t = 2\sin^2 \frac{t}{2}$.
$L = \displaystyle\int_0^{2\pi} 2\left\lvert \sin \frac{t}{2} \right\rvert dt = \Big[-4\cos \frac{t}{2} \Big]_0^{2\pi} = 4 + 4$	$\sin \frac{t}{2} \ge 0$ on $[0, 2\pi]$, so the absolute value can go.
$\text{Area} = \displaystyle\int_0^{2\pi} y\, x'(t)\, dt = \int_0^{2\pi} (1 - \cos t)^2\, dt = \int_0^{2\pi} \left(1 - 2\cos t + \cos^2 t\right) dt$	x increases from 0 to 2π, left to right.
$= 2\pi - 0 + \pi$	$\int_0^{2\pi} \cos^2 t\, dt = \pi$.
Length 8, **area** 3π	The arch is 4 times as long as the wheel's diameter, and its area is 3 times the wheel's.

Problem 6

A point moves along $x = \sin^2 t$, $y = \cos^2 t$ for $0 \le t \le \pi$. How far does it travel, and how long is the curve it traces?

$x + y = 1$, with $0 \le x \le 1$	The point stays on the segment from $(0, 1)$ to $(1, 0)$.
$x' = 2\sin t \cos t = \sin 2t$, $y' = -\sin 2t$, so the speed is $\sqrt{2}\,\lvert\sin 2t\rvert$	$\sqrt{2\sin^2 2t} = \sqrt{2}\,\lvert\sin 2t\rvert$, not $\sqrt{2}\sin 2t$.
$\text{Distance} = \sqrt{2}\displaystyle\int_0^{\pi} \lvert\sin 2t\rvert\, dt = \sqrt{2} \cdot 2$	$\sin 2t$ is negative on $\left(\frac{\pi}{2}, \pi\right)$, so split there: each half contributes 1.

It travels $2\sqrt{2} \approx 2.828$; the segment it traces has length $\sqrt{2}$	At $t = 0$ it is at $(0, 1)$, at $t = \frac{\pi}{2}$ at $(1, 0)$, and at $t = \pi$ back at $(0, 1)$: the segment is covered twice. The arc length integral measures distance traveled, which is the length of the curve only if it is traced once.

Problem 7

> Show that $r = 2\cos\theta$ is a circle, and find its area with a polar integral.

$r^2 = 2r\cos\theta$, so $x^2 + y^2 = 2x$, which is $(x-1)^2 + y^2 = 1$	Multiply by r, then convert. A circle of radius 1 centered at $(1, 0)$.
The circle is traced once as θ runs from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$	$r = 0$ at both ends, and $r \geq 0$ in between.
$\text{Area} = \dfrac{1}{2}\displaystyle\int_{-\pi/2}^{\pi/2} 4\cos^2\theta\, d\theta = \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta)\, d\theta$	$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$.
π	The $\cos 2\theta$ part integrates to 0. A circle of radius 1 has area π, as it should.

Problem 8

> Find the area enclosed by the cardioid $r = 1 + \cos\theta$, and its length.

$\text{Area} = \dfrac{1}{2}\displaystyle\int_0^{2\pi} (1 + \cos\theta)^2\, d\theta = \frac{1}{2}\int_0^{2\pi} \left(1 + 2\cos\theta + \cos^2\theta\right) d\theta$	The cardioid is traced once as θ goes around once, and $r \geq 0$ throughout.
$= \frac{1}{2}(2\pi + 0 + \pi) = \frac{3\pi}{2}$	$\int_0^{2\pi} \cos^2\theta\, d\theta = \pi$.
$r^2 + r'^2 = (1 + \cos\theta)^2 + \sin^2\theta = 2 + 2\cos\theta = 4\cos^2\frac{\theta}{2}$	The half-angle identity again.

$L = \int_0^{2\pi} 2\left\lvert \cos\frac{\theta}{2} \right\rvert d\theta = 2\int_0^{\pi} 2\cos\frac{\theta}{2}\,d\theta = 2\left[4\sin\frac{\theta}{2}\right]_0^{\pi}$	$\cos\frac{\theta}{2} < 0$ for $\pi < \theta < 2\pi$, so use the symmetry about the x-axis and double the top half.
Area $\frac{3\pi}{2} \approx 4.712$, **length** 8	Without the absolute value the length integral would be $\left[4\sin\frac{\theta}{2}\right]_0^{2\pi} = 0$.

Problem 9

> Find the area of the region inside $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$.

$3\sin\theta = 1 + \sin\theta$ gives $\sin\theta = \frac{1}{2}$, so $\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$	Where the curves cross. Between them, $3\sin\theta$ is the outer curve.
$\text{Area} = \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(9\sin^2\theta - (1+\sin\theta)^2\right) d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(8\sin^2\theta - 2\sin\theta - 1\right) d\theta$	Outer radius squared minus inner radius squared.
$\int_{\pi/6}^{5\pi/6} \sin^2\theta\,d\theta = \frac{\pi}{3} + \frac{\sqrt{3}}{4}$, $\int_{\pi/6}^{5\pi/6} \sin\theta\,d\theta = \sqrt{3}$, $\int_{\pi/6}^{5\pi/6} d\theta = \frac{2\pi}{3}$	$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ for the first.
$\frac{1}{2}\left(\frac{8\pi}{3} + 2\sqrt{3} - 2\sqrt{3} - \frac{2\pi}{3}\right)$	The $\sqrt{3}$ terms cancel.
π	The circle $r = 3\sin\theta$ has area $\frac{9\pi}{4}$, so the region is a little under half of it.

Problem 10

> Find the slope of the cardioid $r = 1 + \cos\theta$ at $\theta = \frac{\pi}{2}$.

$r = 1$ and $r' = -\sin\theta = -1$ at $\theta = \frac{\pi}{2}$	The point is $(x, y) = (0, 1)$.

$\dfrac{dy}{d\theta} = r' \sin\theta + r\cos\theta = -1 + 0 = -1$	The product rule on $y = r\sin\theta$.
$\dfrac{dx}{d\theta} = r' \cos\theta - r\sin\theta = 0 - 1 = -1$	The product rule on $x = r\cos\theta$.
$\dfrac{dy}{dx} = \dfrac{-1}{-1} = 1$	As θ increases past $\frac{\pi}{2}$ the point moves left and down, along a line of slope 1.

Where this goes wrong

1. Stopping at d/dt of dy/dx

The first derivative is a ratio of t-derivatives, so the second seems to be one more t-derivative.

$\dfrac{dy}{dx} = \dfrac{3}{2}t - \dfrac{3}{2t}$	Right so far: Problem 2, step 1.
"$\dfrac{d^2y}{dx^2}$ is the derivative of this."	The habit that causes the mistake: the derivative taken with respect to t, and treated as if it were with respect to x.
$\dfrac{d^2y}{dx^2} = \dfrac{3}{2} + \dfrac{3}{2t^2}$, which is $\dfrac{15}{8}$ at $t = 2$ ✗	That is $\frac{d}{dt}\left(\frac{dy}{dx}\right)$. Changing to x needs one more division by $\frac{dx}{dt} = 2t$, and the answer is $\frac{15}{32}$, a quarter of this.

2. Dropping the absolute value in the speed

$\sqrt{a^2} = a$ looks like an ordinary simplification.

$x'^2 + y'^2 = 2\sin^2 2t$ for $x = \sin^2 t$, $y = \cos^2 t$	Right so far: Problem 6, step 2.
"The speed is $\sqrt{2}\sin 2t$."	The habit that causes the mistake: $\sqrt{a^2} = \lvert a\rvert$, and $\sin 2t$ is negative for half of the interval.
Distance $= \sqrt{2}\displaystyle\int_0^{\pi} \sin 2t\, dt = 0$ ✗	A point that moves the whole time cannot travel a distance of 0. The trip out cancels the trip back because the negative speed counts backward. With $\lvert\sin 2t\rvert$, the distance is $2\sqrt{2}$.

3. Integrating over a curve traced twice

A full turn, 0 to 2π, is the default range for θ.

$r = 2\cos\theta$ and area $= \frac{1}{2}\int r^2\,d\theta$	Right so far: Problem 7.
"Integrate from 0 to 2π."	The habit that causes the mistake: assuming every polar curve takes a full turn to trace once.
Area $= \dfrac{1}{2}\displaystyle\int_0^{2\pi} 4\cos^2\theta\,d\theta = 2\pi$ ✗	That is twice the area of a circle of radius 1. For $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, r is negative and the same circle is traced a second time. Find the θ-range that traces the curve once, here $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, and the area is π.

4. Squaring the difference of the radii

The area between two curves in x is the integral of top minus bottom, so the polar version gets written the same way.

Outer $r = 3\sin\theta$, inner $r = 1 + \sin\theta$, for $\frac{\pi}{6} \le \theta \le \frac{5\pi}{6}$	Right so far: Problem 9, step 1.
"Area $= \frac{1}{2}\int (r_{\text{out}} - r_{\text{in}})^2\,d\theta$."	The habit that causes the mistake: the difference taken before squaring.
$\dfrac{1}{2}\displaystyle\int_{\pi/6}^{5\pi/6} (2\sin\theta - 1)^2\,d\theta = \pi - \frac{3\sqrt{3}}{2} \approx 0.544$ ✗	That is the area of wedges of length $r_{\text{out}} - r_{\text{in}}$ starting at the origin, not of the region between the curves. A wedge's area grows with the square of its radius, so subtract the two wedge areas: $\frac{1}{2}\int (r_{\text{out}}^2 - r_{\text{in}}^2)\,d\theta = \pi$.

5. Taking dr/dθ as the slope

For $y = f(x)$ the slope is the derivative, so for $r = f(\theta)$ it seems to be r'.

$r = 1 + \cos\theta$ at $\theta = \frac{\pi}{2}$, the point $(0, 1)$	Right so far: Problem 10, step 1.

"The slope is $\dfrac{dr}{d\theta}$."	The habit that causes the mistake: r and θ treated like y and x.
Slope $= -\sin\frac{\pi}{2} = -1$ ✗	$\frac{dr}{d\theta}$ is how fast the distance from the origin changes, not the steepness of the curve. The slope needs x and y: $\frac{dy/d\theta}{dx/d\theta} = 1$, the opposite sign.

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