Matrix exponentials and linear systems of ODEs

Ten problems on matrix exponentials and linear systems of differential equations: solving x′ = Ax from the eigenvectors and the initial condition, e^(At) = Ve^(Λt)V⁻¹ for a 2×2 matrix, a repeated eigenvalue with one eigenvector and e^(At) = e^(λt)(I + Nt), the rotation e^(Jt) from the series and a decaying spiral, classifying four equilibria from the trace and determinant, a second-order equation rewritten as a first-order system, det e^(At) = e^(t tr A), two matrices with e^(A+B) ≠ e^A e^B, a two-state machine with rate matrix Q and P(t) = e^(Qt), and gradient flow on a quadratic loss compared with gradient descent, with worked solutions and the mistakes of exponentiating entry by entry, splitting e^(A+B) when A and B do not commute, writing V⁻¹e^(Λt)V, treating a defective matrix as diagonalizable and reading stability off the diagonal.

BEFORE YOU START

A single linear differential equation $x' = ax$ has the solution $x(t) = e^{at}x(0)$. A system of them, $x' = Ax$ with a vector x and a square matrix A, has exactly the same solution, $x(t) = e^{At}x(0)$, once the exponential of a matrix is defined by the same power series. Computing e^{At} is an eigenvalue problem: each eigenvector of A grows or decays at its own rate $e^{\lambda t}$, complex eigenvalues turn the motion into a rotation, and the signs of the real parts decide whether every solution dies out. The same object turns a rate matrix into the transition probabilities of a continuous-time Markov chain, and it is what gradient descent approximates when the step size is small. These ten problems compute e^{At} in each of the cases that come up by hand, and the places where the scalar rules stop working for matrices.

- The **matrix exponential** is $e^{M} = I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \cdots$, which converges for every square matrix. It satisfies $\frac{d}{dt}e^{At} = Ae^{At}$ and $e^{A\cdot 0} = I$, so $x(t) = e^{At}x(0)$ solves $x' = Ax$.
- **Diagonalizable case.** If $Av_i = \lambda_i v_i$ and V has the eigenvectors as columns, then $A = V\Lambda V^{-1}$ and $e^{At} = Ve^{\Lambda t}V^{-1}$, where $e^{\Lambda t}$ is diagonal with entries $e^{\lambda_i t}$. Equivalently $x(t) = \sum_i c_i e^{\lambda_i t}v_i$, with the constants $c = V^{-1}x(0)$.
- **Commuting matrices.** If $AB = BA$ then $e^{A+B} = e^{A}e^{B}$. In particular $e^{(\lambda I + N)t} = e^{\lambda t}e^{Nt}$, and if $N^2 = 0$ the series stops: $e^{Nt} = I + Nt$. Without $AB = BA$ the product rule fails.
- **Complex eigenvalues** $\alpha \pm \beta i$ of a real matrix give solutions $e^{\alpha t}$ times a rotation at angular speed β: a spiral, inward if $\alpha < 0$.

- **Stability.** Every solution tends to 0 exactly when every eigenvalue has negative real part. For a 2×2 matrix with trace τ and determinant Δ the eigenvalues solve $\lambda^2 - \tau\lambda + \Delta = 0$: $\Delta < 0$ is a saddle; with $\Delta > 0$ the origin is stable if $\tau < 0$ and unstable if $\tau > 0$, a center if $\tau = 0$, and a spiral when $\tau^2 < 4\Delta$.
- $\det e^{A} = e^{\operatorname{tr} A}$, so e^{A} is never singular, and its inverse is e^{-A}.

Builds on: Eigenvalues and eigenvectors by hand, Second-order linear differential equations with constant coefficients

Problems

1. Solve $x' = Ax$ with $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ and $x(0) = (2, -1)$, using the eigenvalues and eigenvectors of A. difficulty ·

2. For the matrix A of Problem 1, compute $e^{At} = V e^{\Lambda t} V^{-1}$, and check that it reproduces the solution of Problem 1. difficulty ··

3. Show that $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$ has a repeated eigenvalue with only one independent eigenvector. Compute e^{At} by writing $A = 2I + N$, and solve $x' = Ax$ with $x(0) = (1, 0)$. difficulty ··

4. Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Compute e^{Jt} from the power series. Then solve $x' = Bx$ for $B = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix}$ with $x(0) = (1, 0)$, and find $x(\pi/2)$ and $|x(t)|$. difficulty ··

5. Classify the equilibrium at the origin of $x' = Ax$ from the trace and determinant, and confirm with the eigenvalues, for difficulty ·

$$A_1 = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}.$$

6. Rewrite $y'' + 3y' + 2y = 0$ with $y(0) = 1$, $y'(0) = 0$ as a first-order system $x' = Ax$ for $x = (y, y')$, and solve it with eigenvectors. difficulty ··

7. Check that $\det e^{At} = e^{t \operatorname{tr} A}$ for the matrices of Problems 2 and 3, and explain why e^{At} is always invertible. difficulty ·

8. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Compute e^A, e^B, $e^A e^B$, $e^B e^A$ and e^{A+B}, and explain the difference. difficulty ···

9. A machine is up (U) or down (D). It fails at rate 1 per day and is repaired at rate 3 per day, so its rate matrix is $Q = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$ and its transition matrix is $P(t) = e^{Qt}$. Find $P(t)$, the probability that a machine up now is down 12 hours from now, and the long-run fraction of time it is up. difficulty ··

10. Gradient flow on the loss $L(w) = \tfrac{1}{2} w^{\top} H w$ with $H = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ is $w' = -Hw$. Starting from $w(0) = (2, 0)$, find $w(t)$, $L(w(t))$ and when L first drops below 0.01. Then compare gradient descent $w_{k+1} = (I - \eta H) w_k$ with $\eta = 0.1$ for 10 steps against the flow at $t = 1$, and find the step sizes for which gradient descent converges. difficulty ···

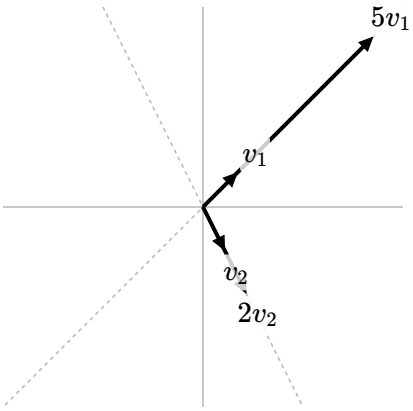
Answers

1. $x(t) = \left(e^{5t} + e^{2t},\ e^{5t} - 2e^{2t}\right)$
2. e^{At} as in step 3; $e^{At}(2,-1) = (e^{5t} + e^{2t},\ e^{5t} - 2e^{2t})$
3. $x(t) = e^{2t}(1 + t,\ -t)$
4. $x(\pi/2) = (-e^{-\pi/2}, 0) \approx (-0.208, 0)$ and $|x(t)| = e^{-t}$
5. A_1 stable node, A_2 center, A_3 saddle, A_4 unstable spiral
6. $y(t) = 2e^{-t} - e^{-2t}$
7. $\det e^{At} = e^{7t}$ and e^{4t}, never 0; the inverse is e^{-At}
8. e^{A+B} equals neither $e^{A}e^{B}$ nor $e^{B}e^{A}$, because $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = BA$
9. $P(t)$ as in step 2; down at 12 hours with probability ≈ 0.216; up $\frac{3}{4}$ of the time
10. $w(t) = e^{-4t}(1,1) + e^{-2t}(1,-1)$, $L = 4e^{-8t} + 2e^{-4t}$, below 0.01 from $t \approx 1.33$; gradient descent converges exactly for $0 < \eta < \frac{1}{2}$

Worked solutions

Problem 1

> Solve $x' = Ax$ with $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ and $x(0) = (2, -1)$, using the eigenvalues and eigenvectors of A.

$\lambda^2 - 7\lambda + 10 = 0$, so $\lambda_1 = 5$, $\lambda_2 = 2$	The trace is 7 and the determinant $12 - 2 = 10$.
$v_1 = (1, 1)$, $v_2 = (1, -2)$ 	$(A - 5I)v = 0$ reads $-x + y = 0$; $(A - 2I)v = 0$ reads $2x + y = 0$.
$c_1(1, 1) + c_2(1, -2) = (2, -1)$ gives $c_1 = 1$, $c_2 = 1$	Subtracting the second equation from the first: $3c_2 = 3$.
$x(t) = e^{5t}(1, 1) + e^{2t}(1, -2)$	Each eigenvector component grows at its own rate $e^{\lambda t}$.
$x(t) = \left(e^{5t} + e^{2t},\ e^{5t} - 2e^{2t}\right)$	Check: $x(0) = (2, -1)$ and $x'(0) = (5 + 2, 5 - 4) = (7, 1) = A\,(2, -1)$. For large t the e^{5t} term dominates and x points along $(1, 1)$.

Problem 2

> For the matrix A of Problem 1, compute $e^{At} = Ve^{\Lambda t}V^{-1}$, and check that it reproduces the solution of Problem 1.

$V = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}, V^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$	The eigenvectors of Problem 1 as columns; $\det V = -3$.
$e^{\Lambda t}V^{-1} = \frac{1}{3}\begin{pmatrix} 2e^{5t} & e^{5t} \\ e^{2t} & -e^{2t} \end{pmatrix}$	A diagonal matrix on the left scales the rows.
$e^{At} = \frac{1}{3}\begin{pmatrix} 2e^{5t}+e^{2t} & e^{5t}-e^{2t} \\ 2e^{5t}-2e^{2t} & e^{5t}+2e^{2t} \end{pmatrix}$	Multiply by V on the left.
At $t=0$ it is I, and its derivative there is $\frac{1}{3}\begin{pmatrix} 12 & 3 \\ 6 & 9 \end{pmatrix} = A$	The two properties that define e^{At}.
e^{At} **as in step 3;** $e^{At}(2,-1) = (e^{5t}+e^{2t},\ e^{5t}-2e^{2t})$	The first entry is $\frac{1}{3}(4e^{5t}+2e^{2t}-e^{5t}+e^{2t})$, matching Problem 1. Writing $V^{-1}e^{\Lambda t}V$ instead gives the transpose of this matrix (Mistake 3).

Problem 3

> Show that $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$ has a repeated eigenvalue with only one independent eigenvector. Compute e^{At} by writing $A = 2I + N$, and solve $x' = Ax$ with $x(0) = (1,0)$.

$\lambda^2 - 4\lambda + 4 = (\lambda-2)^2$	Trace 4, determinant $3+1=4$.
$N = A - 2I = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$, and $Nv = 0$ only for multiples of $(1,-1)$	N has rank 1, so there is one eigenvector and V cannot be built.
$N^2 = 0$, so $e^{Nt} = I + Nt$	The series stops after the linear term.

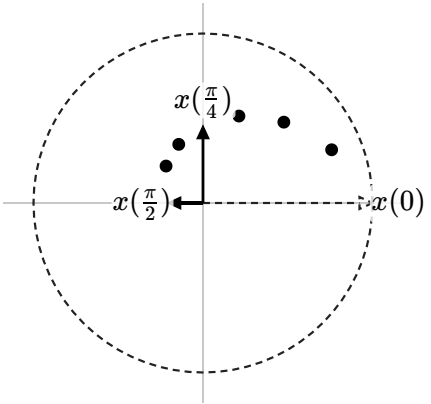
$$\underset{N}{\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}} \cdot \underset{N}{\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}} = \underset{N^2}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}$$

$e^{At} = e^{2t}(I + Nt) = e^{2t}\begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix}$	$2I$ commutes with every matrix, so $e^{(2I+N)t} = e^{2t}e^{Nt}$.
$x(t) = e^{2t}(1+t,\ -t)$	The first column of e^{At}. Check: $x'(0) = (2+1, -1) = (3,-1) = A\,(1,0)$. The factor t next to e^{2t} is what a repeated eigenvalue adds, as te^{rt} does for a repeated characteristic root (Mistake 4).

Problem 4

Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Compute e^{Jt} from the power series. Then solve $x' = Bx$ for $B = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix}$ with $x(0) = (1,0)$, and find $x(\pi/2)$ and $|x(t)|$.

$J^2 = -I$, so $J^3 = -J$ and $J^4 = I$	The powers cycle with period 4, like the powers of i.
$e^{Jt} = \left(1 - \frac{t^2}{2!} + \cdots\right)I + \left(t - \frac{t^3}{3!} + \cdots\right)J$	Group the even and the odd powers.
$e^{Jt} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$	The two series are $\cos t$ and $\sin t$: rotation by the angle t.
$B = -I + 2J$, so $e^{Bt} = e^{-t}e^{2Jt}$	$-I$ commutes with J. The eigenvalues of B are $-1 \pm 2i$.
$x(t) = e^{-t}(\cos 2t,\ \sin 2t)$	The first column of e^{-t} times the rotation by $2t$.



$x(\pi/2) = (-e^{-\pi/2}, 0) \approx (-0.208, 0)$ **and** $\lvert x(t)\rvert = e^{-t}$	A spiral that turns at angular speed 2, the imaginary part of the eigenvalues, and shrinks at rate 1, the negative of the real part. By $t = \frac{\pi}{2}$ it has turned through the angle π, halfway around.

Problem 5

Classify the equilibrium at the origin of $x' = Ax$ from the trace and determinant, and confirm with the eigenvalues, for

$$A_1 = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}.$$

A_1: $\tau = -6$, $\Delta = 8$, $\tau^2 - 4\Delta = 4 > 0$	Real eigenvalues $\frac{-6 \pm 2}{2} = -2, -4$: a stable node.
A_2: $\tau = 0$, $\Delta = -1 + 5 = 4$	Eigenvalues $\pm 2i$: a center. Orbits are closed curves that neither shrink nor grow.
A_3: $\Delta = -6 < 0$	Eigenvalues 3 and -2, of opposite signs: a saddle, unstable.
A_4: $\tau = 4$, $\Delta = 13$, $\tau^2 - 4\Delta = -36 < 0$	Eigenvalues $2 \pm 3i$: an unstable spiral.
A_1 **stable node,** A_2 **center,** A_3 **saddle,** A_4 **unstable spiral**	Only A_1 sends every solution to 0. A_4 has $\Delta > 0$ like A_1, so the determinant alone cannot decide stability; and a matrix with a negative diagonal can still be a saddle (Mistake 5).

Problem 6

Rewrite $y'' + 3y' + 2y = 0$ with $y(0) = 1$, $y'(0) = 0$ as a first-order system $x' = Ax$ for $x = (y, y')$, and solve it with eigenvectors.

$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$, $x(0) = (1, 0)$	The first row says $y' = y'$; the second is the equation solved for y''.

$\det(A-\lambda I)=\lambda^2+3\lambda+2$, so $\lambda=-1,-2$	The characteristic polynomial of A is the characteristic equation of the ODE.
$v_1=(1,-1)$, $v_2=(1,-2)$	For this companion matrix the eigenvector for λ is $(1,\lambda)$: $y=e^{\lambda t}$ has $y'=\lambda e^{\lambda t}$.
$c_1+c_2=1$, $-c_1-2c_2=0$, so $c_1=2$, $c_2=-1$	Match $x(0)=(1,0)$.
$y(t)=2e^{-t}-e^{-2t}$	The first entry of $2e^{-t}(1,-1)-e^{-2t}(1,-2)$. Check: $y(0)=1$ and $y'(0)=-2+2=0$. Both eigenvalues are negative and real, so y decays without oscillating: the system is overdamped.

Problem 7

> Check that $\det e^{At}=e^{t\,\mathrm{tr}\,A}$ for the matrices of Problems 2 and 3, and explain why e^{At} is always invertible.

Problem 2: $9\det e^{At}=(2e^{5t}+e^{2t})(e^{5t}+2e^{2t})-2(e^{5t}-e^{2t})^2$	The factor $\frac{1}{3}$ squared comes out as $\frac{1}{9}$.
$=(2e^{10t}+5e^{7t}+2e^{4t})-(2e^{10t}-4e^{7t}+2e^{4t})=9e^{7t}$	So $\det e^{At}=e^{7t}$, and $\mathrm{tr}\,A=7$.
Problem 3: $\det e^{At}=e^{4t}\big((1+t)(1-t)+t^2\big)=e^{4t}$	Pulling e^{2t} out of both rows gives e^{4t}; the bracket is 1. $\mathrm{tr}\,A=4$.
In general $\det e^{At}=\prod_i e^{\lambda_i t}=e^{t\sum_i \lambda_i}$	The eigenvalues of e^{At} are $e^{\lambda_i t}$, and the determinant is their product.
$\det e^{At}=e^{7t}$ **and** e^{4t}**, never** 0**; the inverse is** e^{-At}	At and $-At$ commute, so $e^{At}e^{-At}=e^{0}=I$. Running the system backward in time undoes it.

Problem 8

Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Compute e^A, e^B, $e^A e^B$, $e^B e^A$ and e^{A+B}, and explain the difference.

$e^A = I + A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $e^B = I + B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$A^2 = B^2 = 0$, so each series stops.
$e^A e^B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $e^B e^A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$	Already the two orders disagree.
$\underset{e^A}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} \cdot \underset{e^B}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad$ $\underset{e^B}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}} \cdot \underset{e^A}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$	
$(A+B)^2 = I$, so $e^{A+B} = (\cosh 1)I + (\sinh 1)(A+B)$	Even powers are I and odd powers are $A+B$, like e^J in Problem 4 without the signs.
$e^{A+B} = \begin{pmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{pmatrix} \approx \begin{pmatrix} 1.543 & 1.175 \\ 1.175 & 1.543 \end{pmatrix}$	Symmetric, unlike either product.
e^{A+B} **equals neither** $e^A e^B$ **nor** $e^B e^A$, **because** $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = BA$	The step $e^a e^b = e^{a+b}$ rearranges as past b s, which matrices allow only when they commute. All three matrices have determinant $1 = e^{\mathrm{tr}(A+B)}$, so the determinant cannot catch the error (Mistake 2).

Problem 9

A machine is up (U) or down (D). It fails at rate 1 per day and is repaired at rate 3 per day, so its rate matrix is $Q = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$ and its transition matrix is $P(t) = e^{Qt}$. Find $P(t)$, the

probability that a machine up now is down 12 hours from now, and the long-run fraction of time it is up.

The eigenvalues of Q are 0 and -4	Each row of Q sums to 0, so $Q\mathbf{1} = 0$; the trace gives the other, -4.
$P(t) = \frac{1}{4}\begin{pmatrix} 3 + e^{-4t} & 1 - e^{-4t} \\ 3 - 3e^{-4t} & 1 + 3e^{-4t} \end{pmatrix}$	The two-state formula of the Markov chains set with λ_2^n replaced by e^{-4t}; at $t = 0$ it is I, and its derivative there is Q.
$\Pr(D \text{ at } t = \frac{1}{2} \mid U) = \frac{1}{4}(1 - e^{-2}) \approx 0.216$	Twelve hours is $t = \frac{1}{2}$ day, so $e^{-4t} = e^{-2} \approx 0.135$.
As $t \to \infty$, both rows tend to $(\frac{3}{4}, \frac{1}{4})$	The stationary distribution solves $\pi Q = 0$: $-\pi_U + 3\pi_D = 0$.
$P(t)$ as in step 2; down at 12 hours with probability ≈ 0.216; up $\frac{3}{4}$ of the time	The mean up time is 1 day and the mean repair $\frac{1}{3}$ day, and $1/(1 + \frac{1}{3}) = \frac{3}{4}$. The other entry of the top row at $t = \frac{1}{2}$ is ≈ 0.784, and each row sums to 1, which an entrywise exponential does not (Mistake 1).

Problem 10

Gradient flow on the loss $L(w) = \frac{1}{2} w^\top H w$ with $H = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ is $w' = -Hw$. Starting from $w(0) = (2, 0)$, find $w(t)$, $L(w(t))$ and when L first drops below 0.01. Then compare gradient descent $w_{k+1} = (I - \eta H) w_k$ with $\eta = 0.1$ for 10 steps against the flow at $t = 1$, and find the step sizes for which gradient descent converges.

H has eigenvalues 4 and 2 with eigenvectors $(1, 1)$ and $(1, -1)$, and $w(0) = (1, 1) + (1, -1)$	H is symmetric, so its eigenvectors are orthogonal.
$w(t) = e^{-4t}(1, 1) + e^{-2t}(1, -1)$	e^{-Ht} multiplies each eigenvector component by $e^{-\lambda t}$.
$L(w(t)) = \frac{1}{2}\left(4 \cdot 2e^{-8t} + 2 \cdot 2e^{-4t}\right) = 4e^{-8t} + 2e^{-4t}$	Orthogonal components add in $w^\top H w$; each $(1, \pm 1)$ has squared length 2. At $t = 0$, $L = 6$.

With $u = e^{-4t}$: $4u^2 + 2u = 0.01$ gives $u \approx 0.00495$ and $t = -\frac{1}{4}\ln u \approx 1.33$	The positive root of the quadratic. The slow direction, eigenvalue 2, sets the time.
Gradient descent multiplies the components by $1 - 0.1 \cdot 4 = 0.6$ and $1 - 0.1 \cdot 2 = 0.8$ per step	$(I - \eta H)$ has the same eigenvectors as H.
After 10 steps: $0.6^{10} \approx 0.00605$ and $0.8^{10} \approx 0.107$, against $e^{-4} \approx 0.0183$ and $e^{-2} \approx 0.135$	Each step is the first two terms of $e^{-\eta H}$; with $\eta = 0.1$ that is close for the slow direction and rough for the fast one.
$w(t) = e^{-4t}(1,1) + e^{-2t}(1,-1)$, $L = 4e^{-8t} + 2e^{-4t}$, **below** 0.01 **from** $t \approx 1.33$; **gradient descent converges exactly for** $0 < \eta < \frac{1}{2}$	Convergence needs $\lvert 1 - \eta\lambda \rvert < 1$ for both eigenvalues, so $\eta < 2/\lambda_{\max} = \frac{1}{2}$. The flow converges for every t; only the discrete steps can overshoot.

Where this goes wrong

1. Exponentiating entry by entry

The scalar exponential is applied to each entry, as `np.exp` does to an array.

$Q = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$, $t = \frac{1}{2}$	Right so far: Problem 9.
"$P(\frac{1}{2}) = e^{Qt}$, so exponentiate each entry of $\frac{1}{2}Q$."	e^M is the power series in the matrix M; its entries mix all the entries of M. In Python it is `scipy.linalg.expm`, not numpy's `np.exp`.
$P(\frac{1}{2}) = \begin{pmatrix} e^{-0.5} & e^{0.5} \\ e^{1.5} & e^{-1.5} \end{pmatrix}$ ✗	Its first row is $(0.607, 1.649)$, which sums to 2.255, and at $t = 0$ the same rule gives a matrix of ones instead of I. The correct first row is $(0.784, 0.216)$.

2. Splitting e^(A+B) when A and B do not commute

The scalar rule $e^{a+b} = e^a e^b$ is used to split a matrix into easy pieces.

$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = D + N$ with $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	Right so far: a diagonal part and a part with $N^2 = 0$.
"$e^{At} = e^{Dt}e^{Nt} = e^{Dt}(I + Nt)$."	$DN = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ but $ND = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, so the split is not allowed. It worked in Problems 3 and 4 because the diagonal part there was a multiple of I.
$e^A = \begin{pmatrix} e & e \\ 0 & e^2 \end{pmatrix}$ ✗	The eigenvectors $(1, 0)$ and $(1, 1)$ give $e^{At} = \begin{pmatrix} e^t & e^{2t} - e^t \\ 0 & e^{2t} \end{pmatrix}$, so the corner of e^A is $e^2 - e \approx 4.671$, not $e \approx 2.718$.

3. V⁻¹ e^(Λt) V instead of V e^(Λt) V⁻¹

The two outer factors are written in the wrong order.

$V = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$, $V^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$ for $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$	Right so far: Problem 2.
"$e^{At} = V^{-1}e^{\Lambda t}V$."	$A = V\Lambda V^{-1}$: V^{-1} converts x into eigenvector coordinates, $e^{\Lambda t}$ scales them, and V converts back, so V^{-1} acts first and sits on the right.
$e^{At} = \frac{1}{3}\begin{pmatrix} 2e^{5t} + e^{2t} & 2e^{5t} - 2e^{2t} \\ e^{5t} - e^{2t} & e^{5t} + 2e^{2t} \end{pmatrix}$ ✗	Its derivative at $t = 0$ is $\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} = A^\top$, not A: here it is the transpose of the right answer, because this V happens to be symmetric. In general $V^{-1}e^{\Lambda t}V$ is the exponential of $V^{-1}\Lambda V$, a different matrix with the same eigenvalues, so a check of the trace or the determinant cannot catch the slip.

4. A repeated eigenvalue treated as two eigenvectors

The eigenvalue 2 is listed twice and V is assumed to exist.

$A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$ has the double eigenvalue 2	Right so far: Problem 3.
"So $\Lambda = 2I$ and $e^{At} = Ve^{2tI}V^{-1} = e^{2t}I$."	That would make $A = V(2I)V^{-1} = 2I$. A matrix that is not already $2I$ has only one eigenvector here, and V is singular.
$x(t) = e^{2t}(1, 0)$ for $x(0) = (1, 0)$ ✗	Its derivative at 0 is $(2, 0)$, but $A(1, 0) = (3, -1)$. The correct solution is $e^{2t}(1 + t, -t)$, with the extra t from $e^{Nt} = I + Nt$.

5. Stability read off the diagonal

Each entry on the diagonal is treated as a decay rate, as if the coordinates were separate equations.

$x' = Ax$ with $A = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}$	Right so far.
"$x_1' = -x_1 + \ldots$ and $x_2' = -x_2 + \ldots$: both coordinates decay."	The off-diagonal terms couple the coordinates, and stability is set by the eigenvalues. Here $\Delta = 1 - 9 = -8 < 0$.
Every solution tends to 0 ✗	The eigenvalues are 2 and -4: a saddle. From $x(0) = (1, 1)$, an eigenvector for 2, the solution is $e^{2t}(1, 1)$, already $e^2 \approx 7.39$ times as large at $t = 1$.

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