Linear first-order differential equations: the integrating factor

Ten linear first-order differential equations solved with an integrating factor: a constant forcing term, e^(2x) on the right, xy′ + 2y = x² after rewriting in standard form, a forcing term that needs integration by parts, sec x as the factor, one that is also separable, a salt tank with constant volume, a sinusoidal input and its lagging steady state, a tank that fills while it drains, and the Bernoulli substitution that turns the logistic equation into a linear one, with worked solutions and the mistakes of multiplying only one side, adding the constant after dividing and holding a changing volume constant.

BEFORE YOU START

A first-order equation is linear when y and y' appear only to the first power, with coefficients that may depend on x: $y' + P(x)\,y = Q(x)$. Most of these are not separable, because the right side is a sum rather than a product, but all of them can be solved the same way. Multiply by an integrating factor that turns the left side into the derivative of a product, then integrate. Mixing tanks, circuits, cooling with a changing room temperature and the continuous-time version of an exponential moving average are all equations of this kind. The work goes wrong in a few places: the factor computed before the equation is in standard form or with the wrong sign, only the left side multiplied, the constant added after dividing by the factor, and a changing volume treated as fixed. These ten problems cover each of them.

- **Standard form.** Divide by the coefficient of y' so the equation reads $y' + P(x)\,y = Q(x)$. P is the coefficient of y **after** this division, sign included.
- **Integrating factor.** $\mu(x) = e^{\int P(x)\,dx}$, with no constant needed. It is chosen so that $\mu' = P\mu$, and then $\mu y' + P\mu y = (\mu y)'$ by the product rule.
- **The solution.** Multiply **both** sides by μ: $(\mu y)' = \mu Q$. Integrate, $\mu y = \displaystyle\int \mu Q\,dx + C$, and only then divide by μ: $y = \dfrac{1}{\mu}\Big(\displaystyle\int \mu Q\,dx + C\Big)$. The constant ends up multiplied by $\frac{1}{\mu}$.
- **Structure.** The solution is one particular solution plus $\frac{C}{\mu}$, the general solution of $y' + Py = 0$. When P is a positive constant, $\frac{C}{\mu} = Ce^{-Px}$ is a transient that dies out.
- **Mixing problems.** If $Q(t)$ is the amount of salt and $V(t)$ the volume, $Q' = (\text{rate in}) - \dfrac{Q}{V(t)} \cdot (\text{outflow rate})$. V is constant only if inflow and outflow are equal.
- **Bernoulli equations.** $y' + Py = Qy^n$ becomes linear in $v = y^{1-n}$.

> Builds on: Separable differential equations: growth, decay, cooling and the logistic curve, Integration: substitution and parts

Problems

1. Solve $y' + 2y = 6$ with $y(0) = 1$. difficulty ·

2. Solve $y' - y = e^{2x}$ with $y(0) = 0$. difficulty ·

3. Solve $xy' + 2y = x^2$ for $x > 0$, with $y(1) = 1$. difficulty ··

4. Solve $y' + y = x$ with $y(0) = 0$. difficulty ··

5. Solve $y' + (\tan x)\, y = \cos x$ on $-\frac{\pi}{2} < x < \frac{\pi}{2}$, with $y(0) = 1$. difficulty ··

6. Solve $y' + 2xy = x$ with $y(0) = 0$, with an integrating factor and then by separating. difficulty ··

7. A tank holds 100 L of water with 10 kg of salt dissolved in it. Brine with 0.2 kg of salt per liter flows in at 5 L/min, and the well-mixed solution flows out at 5 L/min. Find the salt $Q(t)$, its value at $t = 20$, and when it reaches 19 kg. difficulty ··

8. A first-order filter, the continuous version of an exponential moving average, obeys $y' + y = \sin x$, with $y(0) = 0$. Solve it, and describe the solution for large x. difficulty ···

9. A tank starts with 20 L of pure water. Brine with 3 g of salt per liter flows in at 2 L/min, and the mixture flows out at 1 L/min. Find the salt $Q(t)$, and the amount and concentration at $t = 20$ min. difficulty ···

10. Solve the logistic equation $y' = y - y^2$ with $y(0) = \frac{1}{2}$ by the substitution $v = \frac{1}{y}$. difficulty ···

Answers

1. $y = 3 - 2e^{-2x}$
2. $y = e^{2x} - e^{x}$
3. $y = \dfrac{x^2}{4} + \dfrac{3}{4x^2}$
4. $y = x - 1 + e^{-x}$
5. $y = (x + 1)\cos x$
6. $y = \frac{1}{2}\left(1 - e^{-x^2}\right)$
7. $Q = 20 - 10e^{-t/20}$; $Q(20) \approx 16.32$ kg; 19 kg at $t = 20\ln 10 \approx 46.05$ min
8. $y = \frac{1}{2}(\sin x - \cos x) + \frac{1}{2}e^{-x}$; for large x, $y \approx \frac{1}{\sqrt{2}}\sin\left(x - \frac{\pi}{4}\right)$
9. $Q = 3(20 + t) - \dfrac{1200}{20 + t}$; at $t = 20$, 90 g in 40 L, or 2.25 g/L
10. $y = \dfrac{1}{1 + e^{-x}}$, the sigmoid

Worked solutions

Problem 1

> Solve $y' + 2y = 6$ with $y(0) = 1$.

$\mu = e^{\int 2\,dx} = e^{2x}$	Already in standard form, with $P = 2$.
$(e^{2x}y)' = 6e^{2x}$	Multiply both sides by μ; the left side is now a product's derivative.
$e^{2x}y = 3e^{2x} + C$	Integrate both sides.
$y = 3 + Ce^{-2x}$, and $y(0) = 3 + C = 1$ gives $C = -2$	Divide by μ, then use the initial condition.
$y = 3 - 2e^{-2x}$	Check: $y' = 4e^{-2x}$ and $y' + 2y = 4e^{-2x} + 6 - 4e^{-2x} = 6$. The solution rises toward the equilibrium $y = 3$.

Problem 2

> Solve $y' - y = e^{2x}$ with $y(0) = 0$.

$P = -1$, so $\mu = e^{-x}$	The sign of P goes into the exponent.
$(e^{-x}y)' = e^{-x}e^{2x} = e^{x}$	Multiply both sides by e^{-x}.
$e^{-x}y = e^{x} + C$, so $y = e^{2x} + Ce^{x}$	Integrate, then divide by e^{-x}, which multiplies by e^{x}.
$y(0) = 1 + C = 0$ gives $C = -1$	The initial condition.
$y = e^{2x} - e^{x}$	Check: $y' - y = (2e^{2x} - e^{x}) - (e^{2x} - e^{x}) = e^{2x}$.

Problem 3

> Solve $xy' + 2y = x^2$ for $x > 0$, with $y(1) = 1$.

$y' + \dfrac{2}{x}\,y = x$	Divide by x to reach standard form; now $P = \frac{2}{x}$.
$\mu = e^{\int 2/x\,dx} = e^{2\ln x} = x^2$	For $x > 0$, $\ln\lvert x\rvert = \ln x$.
$(x^2 y)' = x^2 \cdot x = x^3$	Multiply the standard form by x^2. This is the original equation multiplied by x.
$x^2 y = \dfrac{x^4}{4} + C$, and $1 = \frac{1}{4} + C$ gives $C = \frac{3}{4}$	Integrate, then use $y(1) = 1$.
$y = \dfrac{x^2}{4} + \dfrac{3}{4x^2}$	Check: $xy' = \frac{x^2}{2} - \frac{3}{2x^2}$ and $2y = \frac{x^2}{2} + \frac{3}{2x^2}$, which add to x^2. For example $y(2) = 1 + \frac{3}{16} = \frac{19}{16}$.

Problem 4

> Solve $y' + y = x$ with $y(0) = 0$.

$\mu = e^x$, so $(e^x y)' = xe^x$	$P = 1$.
$\displaystyle\int xe^x\,dx = xe^x - e^x$	By parts with $u = x$, $dv = e^x\,dx$.
$e^x y = (x-1)e^x + C$, so $y = x - 1 + Ce^{-x}$	Add the constant before dividing by e^x.
$y(0) = -1 + C = 0$ gives $C = 1$	The initial condition.
$y = x - 1 + e^{-x}$	Check: $y' + y = (1 - e^{-x}) + (x - 1 + e^{-x}) = x$. For large x the solution follows the line $y = x - 1$, one unit behind the input.

Problem 5

Solve $y' + (\tan x)\, y = \cos x$ on $-\frac{\pi}{2} < x < \frac{\pi}{2}$, with $y(0) = 1$.

$\int \tan x \, dx = -\ln\lvert\cos x\rvert$, so $\mu = e^{-\ln\cos x} = \dfrac{1}{\cos x} = \sec x$	$\cos x > 0$ on this interval.
$(y \sec x)' = \cos x \sec x = 1$	Multiply both sides by $\sec x$.
$y \sec x = x + C$, so $y = (x + C)\cos x$	Integrate, then multiply by $\cos x$.
$y = (x + 1)\cos x$	$y(0) = C = 1$. Check: $y' = \cos x - (x + 1)\sin x$ and $y\tan x = (x + 1)\sin x$, which add to $\cos x$.

Problem 6

Solve $y' + 2xy = x$ with $y(0) = 0$, with an integrating factor and then by separating.

$\mu = e^{\int 2x\, dx} = e^{x^2}$, so $(e^{x^2} y)' = x e^{x^2}$	$P = 2x$.
$e^{x^2} y = \frac{1}{2} e^{x^2} + C$, so $y = \frac{1}{2} + C e^{-x^2}$, and $C = -\frac{1}{2}$	$\int x e^{x^2}\, dx = \frac{1}{2} e^{x^2}$ by $u = x^2$.
Separating: $y' = x(1 - 2y)$, so $\dfrac{dy}{1 - 2y} = x\, dx$ and $-\frac{1}{2}\ln\lvert 1 - 2y\rvert = \frac{x^2}{2} + c$	Here the right side factors, so this one is separable as well.
$1 - 2y = A e^{-x^2}$ with $A = 1$ from $y(0) = 0$	The same family.
$y = \frac{1}{2}\left(1 - e^{-x^2}\right)$	Both methods agree, as they must. With $P \neq 0$, an equation $y' + Py = Q$ is separable exactly when $\frac{Q}{P}$ is a constant; here it is $\frac{1}{2}$, the value y settles at.

Problem 7

A tank holds 100 L of water with 10 kg of salt dissolved in it. Brine with 0.2 kg of salt per liter flows in at 5 L/min, and the well-mixed solution flows out at 5 L/min. Find the salt $Q(t)$, its value at $t = 20$, and when it reaches 19 kg.

$Q' = 0.2 \cdot 5 - \dfrac{Q}{100} \cdot 5 = 1 - \dfrac{Q}{20}$	In: 1 kg/min. Out: the concentration $\frac{Q}{100}$ times 5 L/min. The volume stays at 100 L.
$Q' + \dfrac{Q}{20} = 1$, $\mu = e^{t/20}$, $(e^{t/20}Q)' = e^{t/20}$	Standard form with $P = \frac{1}{20}$.
$e^{t/20}Q = 20e^{t/20} + C$, so $Q = 20 + Ce^{-t/20}$, and $Q(0) = 10$ gives $C = -10$	Integrate, divide, apply $Q(0)$.
$Q(20) = 20 - 10e^{-1}$; $Q = 19$ when $10e^{-t/20} = 1$, $t = 20 \ln 10$	The gap to 20 kg shrinks by a factor of e every 20 minutes.
$Q = 20 - 10e^{-t/20}$; $Q(20) \approx 16.32$ **kg;** 19 **kg at** $t = 20 \ln 10 \approx 46.05$ **min**	In the long run the tank holds $100 \times 0.2 = 20$ kg, the inflow concentration.

Problem 8

A first-order filter, the continuous version of an exponential moving average, obeys $y' + y = \sin x$, with $y(0) = 0$. Solve it, and describe the solution for large x.

$\mu = e^{x}$, $(e^{x}y)' = e^{x}\sin x$	$P = 1$.
$\displaystyle\int e^{x}\sin x\,dx = \frac{e^{x}(\sin x - \cos x)}{2}$	By parts twice, solving for the integral when it reappears.
$y = \dfrac{\sin x - \cos x}{2} + Ce^{-x}$, and $y(0) = -\frac{1}{2} + C = 0$ gives $C = \frac{1}{2}$	Divide by e^{x} and apply $y(0) = 0$.
$\dfrac{\sin x - \cos x}{2} = \dfrac{1}{\sqrt{2}}\sin\left(x - \dfrac{\pi}{4}\right)$	$\sin(x - \frac{\pi}{4}) = \frac{\sin x - \cos x}{\sqrt{2}}$.

$y = \frac{1}{2}(\sin x - \cos x) + \frac{1}{2}e^{-x}$; **for large** x, $y \approx \frac{1}{\sqrt{2}} \sin\left(x - \frac{\pi}{4}\right)$	The transient $\frac{1}{2}e^{-x}$ dies out. What is left follows the input with its amplitude cut to $\frac{1}{\sqrt{2}} \approx 0.707$ and a lag of $\frac{\pi}{4}$: smoothing always costs delay, which is the tradeoff in choosing an EMA's decay rate.

Problem 9

> A tank starts with 20 L of pure water. Brine with 3 g of salt per liter flows in at 2 L/min, and the mixture flows out at 1 L/min. Find the salt $Q(t)$, and the amount and concentration at $t =$ 20 min.

$V(t) = 20 + t$	The volume grows by $2 - 1 = 1$ L/min.
$Q' = 3 \cdot 2 - \dfrac{Q}{20 + t} \cdot 1 = 6 - \dfrac{Q}{20 + t}$	The outflow concentration is $\frac{Q}{V(t)}$, with the current volume.
$\mu = e^{\int dt/(20+t)} = 20 + t$, so $\big((20 + t)Q\big)' = 6(20 + t)$	$P = \frac{1}{20+t}$.
$(20 + t)Q = 3(20 + t)^2 + C$, and $Q(0) = 0$ gives $C = -1200$	$\int 6(20 + t)\,dt = 3(20 + t)^2$; at $t = 0$, $0 = 1200 + C$.
$Q = 3(20 + t) - \dfrac{1200}{20 + t}$; **at** $t = 20$, 90 **g in** 40 **L, or** 2.25 **g/L**	$Q(20) = 120 - 30$. The concentration approaches 3 g/L, the inflow's, as the tank grows.

Problem 10

> Solve the logistic equation $y' = y - y^2$ with $y(0) = \frac{1}{2}$ by the substitution $v = \frac{1}{y}$.

$v' = -\dfrac{y'}{y^2} = -\dfrac{y - y^2}{y^2} = 1 - v$	A Bernoulli equation with $n = 2$, so $v = y^{1-2} = \frac{1}{y}$.
$v' + v = 1$, $\mu = e^x$, so $e^x v = e^x + C$ and $v = 1 + Ce^{-x}$	Linear in v, with $P = 1$.

$v(0) = 2$ gives $C = 1$	$v(0) = \frac{1}{y(0)}$.
$y = \dfrac{1}{1 + e^{-x}}$, **the sigmoid**	Check: $y' = \frac{e^{-x}}{(1+e^{-x})^2} = y(1 - y)$. Separating with partial fractions gives the same answer; the substitution avoids the partial fractions.

Where this goes wrong

1. Finding μ before the standard form

The integrating factor is computed from the equation exactly as it is written.

$xy' + 2y = x^2$	Right so far: Problem 3.
"The coefficient of y is 2, so $\mu = e^{\int 2\,dx}$."	The habit that causes the mistake: P read off before dividing by the coefficient of y'.
$\mu = e^{2x}$ ✗	Multiplying by e^{2x} gives $xe^{2x}y' + 2e^{2x}y$, but $(e^{2x}y)' = e^{2x}y' + 2e^{2x}y$, so the left side is not a derivative and nothing can be integrated. Divide by x first: $P = \frac{2}{x}$, $\mu = x^2$, and $y = \frac{x^2}{4} + \frac{3}{4x^2}$, with $y(2) = \frac{19}{16}$.

2. Multiplying only the left side by μ

The factor is there to make the left side a derivative, so it is easy to apply it there and nowhere else.

$\mu = e^{2x}$ and $(e^{2x}y)' = e^{2x}y' + 2e^{2x}y$	Right so far: Problem 1.
"So $(e^{2x}y)' = 6$."	The habit that causes the mistake: the right side left as it was.
$e^{2x}y = 6x + C$, and $y(0) = 1$ gives $y = (6x + 1)e^{-2x}$ ✗	At $x = 1$ this gives $7e^{-2} \approx 0.947$, while the solution is $3 - 2e^{-2} \approx 2.729$. It also decays to 0 rather than settling at 3. The right side must be multiplied as well: $(e^{2x}y)' = 6e^{2x}$.

3. Dropping the sign of P

$\mu = e^{\int P}$ is remembered as an exponential of the coefficient, and its sign is lost.

$y' - y = e^{2x}$	Right so far: Problem 2, where $P = -1$.
"$\mu = e^{x}$."	The habit that causes the mistake: the coefficient taken as 1 instead of -1.
$(e^{x}y)' = e^{3x}$, so $y = \frac{1}{3}\left(e^{2x} - e^{-x}\right)$ ✗	$(e^{x}y)' = e^{x}(y' + y)$, not $e^{x}(y' - y)$. Check the result: at $x = 1$, $y' - y = \frac{1}{3}\left(e^{2} + 2e^{-1}\right) \approx 2.708$, but $e^{2} \approx 7.389$. With $\mu = e^{-x}$ the answer is $y = e^{2x} - e^{x}$.

4. Adding the constant after dividing by μ

The constant of integration is added on the last line, as for a plain antiderivative.

$(e^{x}y)' = xe^{x}$, so $e^{x}y = (x - 1)e^{x}$	Right so far, apart from the constant: Problem 4, step 3.
"Divide by e^{x}, then add C."	The habit that causes the mistake: the constant belongs to the integration, before the division.
$y = x - 1 + C$, and $y(0) = 0$ gives $y = x$ ✗	Check: $y' + y = 1 + x$, not x. The constant has to be divided by μ too: $y = x - 1 + Ce^{-x}$, and the solution is $y = x - 1 + e^{-x}$.

5. Holding a changing volume constant

Mixing problems usually have equal inflow and outflow, so the volume is taken as fixed.

$Q' = 6 - (\text{concentration}) \cdot 1$	Right so far: Problem 9, where 2 L/min flows in and 1 L/min flows out.
"The concentration is $\frac{Q}{20}$."	The habit that causes the mistake: the starting volume used for all time.
$Q' = 6 - \frac{Q}{20}$, so $Q = 120\left(1 - e^{-t/20}\right)$ and $Q(20) \approx 75.85$ g ✗	The tank gains 1 L every minute, so at $t = 20$ it holds 40 L, not 20, and less salt leaves than this assumes. With $V = 20 + t$ the answer is 90 g.

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