L'Hôpital's rule and indeterminate forms

Ten limits by L'Hôpital's rule: 0/0 and ∞/∞, a rule applied twice, x ln x and the 0 log 0 = 0 of entropy, x^x and $(1 + 2/x)^x$ by taking logs, a difference of fractions, and a limit the rule cannot find, with worked solutions and the mistakes of using it on a limit that is not indeterminate and of reading 1^∞ as 1.

BEFORE YOU START

When substitution gives $\frac{0}{0}$ or $\frac{\infty}{\infty}$, L'Hôpital's rule replaces the limit of a ratio with the limit of the ratio of the derivatives. It is how the convention $0 \log 0 = 0$ in entropy is justified, and why $x^2 e^{-x}$ and $\frac{\ln x}{\sqrt{x}}$ go to 0: exponentials beat powers, and powers beat logs. The work goes wrong in a few places: the rule used on a limit that is not indeterminate, the quotient rule used instead of differentiating top and bottom separately, 1^∞ read as 1, a log taken and never undone, and a conclusion drawn when the rule does not apply. These ten problems cover each of them.

- **L'Hôpital's rule.** Suppose f and g are differentiable near a (except possibly at a), $g'(x) \neq 0$ near a, and $\frac{f(x)}{g(x)}$ has the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ as $x \to a$. If $\lim \frac{f'(x)}{g'(x)}$ exists (or is $\pm\infty$), then $\lim \frac{f(x)}{g(x)}$ equals it. The same holds for one-sided limits and for $a = \pm\infty$.
- f' and g' are differentiated **separately**. It is not the quotient rule.
- Check the form before every use. If substitution gives a number, that number is the limit and the rule does not apply.
- The other indeterminate forms are rewritten first. $0 \cdot \infty$: write the product as a quotient, $fg = \frac{f}{1/g}$. $\infty - \infty$: combine into one fraction. 1^∞, 0^0 and ∞^0: take the log, find the limit L of the log, and the answer is e^L.
- If $\lim \frac{f'}{g'}$ does not exist, the rule says nothing: the original limit may still exist (Problem 10).
- Derivatives used: $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(e^x)' = e^x$, $(\ln x)' = \frac{1}{x}$, $(x^n)' = nx^{n-1}$.

Builds on: Evaluating limits: factoring, conjugates and limits at infinity, Differentiation rules: chain, product, quotient

Problems

1. Find $\lim_{x \to 0} \dfrac{\sin 3x}{x}$. difficulty ·

2. Find $\lim_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$. difficulty ·

3. Find $\lim_{x \to \infty} \dfrac{x^2}{e^x}$. difficulty ··

4. Find $\lim_{x \to 0^+} x \ln x$. difficulty ··

5. Find $\lim_{x \to \infty} \dfrac{\ln x}{\sqrt{x}}$. difficulty ··

6. Find $\lim_{x \to 0} \dfrac{1 - \cos x}{x^2}$. difficulty ··

7. Find $\lim_{x \to 0^+} x^x$. difficulty ···

8. Find $\lim_{x \to \infty} \left(1 + \dfrac{2}{x}\right)^x$. difficulty ···

9. Find $\lim_{x \to 0} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$. difficulty ···

10. Find $\lim_{x \to \infty} \dfrac{x + \sin x}{x}$. What happens if you use L'Hôpital's rule? difficulty ···

Answers

1. 3
2. $\frac{1}{2}$
3. 0
4. 0
5. 0
6. $\frac{1}{2}$
7. 1
8. $e^2 \approx 7.389$
9. 0
10. 1; L'Hôpital's rule does not apply, because $\lim(1 + \cos x)$ does not exist

Worked solutions

Problem 1

> Find $\lim_{x \to 0} \dfrac{\sin 3x}{x}$.

At $x = 0$: $\frac{\sin 0}{0} = \frac{0}{0}$.	Indeterminate, so the rule may apply.
$\lim_{x \to 0} \dfrac{3 \cos 3x}{1}$	Differentiate top and bottom separately; the chain rule gives the 3.
3	$3 \cos 0 = 3$. Near 0, $\sin 3x \approx 3x$.

Problem 2

> Find $\lim_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$.

At $x = 0$: $\frac{1-1-0}{0} = \frac{0}{0}$.	Indeterminate.
$\lim_{x \to 0} \dfrac{e^x - 1}{2x}$, again $\frac{0}{0}$	One application. Check the form again before the next one.
$\lim_{x \to 0} \dfrac{e^x}{2}$	A second application.
$\frac{1}{2}$	$\frac{e^0}{2}$. It is the $\frac{x^2}{2}$ term of the Taylor series $e^x = 1 + x + \frac{x^2}{2} + \cdots$.

Problem 3

> Find $\lim_{x \to \infty} \dfrac{x^2}{e^x}$.

As $x \to \infty$: $\frac{\infty}{\infty}$.	Indeterminate.
$\lim_{x \to \infty} \dfrac{2x}{e^x}$, still $\frac{\infty}{\infty}$	The power drops by one; the exponential does not change.

$\lim_{x\to\infty} \dfrac{2}{e^x}$	A second application. The top is now a constant.
0	$e^x \to \infty$. The same argument, applied n times, shows $x^n e^{-x} \to 0$ for every n.

Problem 4

Find $\lim_{x\to 0^+} x \ln x$.

As $x \to 0^+$: $0 \cdot (-\infty)$.	Indeterminate, but not a quotient yet.
$x \ln x = \dfrac{\ln x}{1/x}$, which has the form $\frac{-\infty}{\infty}$.	Move x to the denominator as $\frac{1}{x}$. Keeping $\ln x$ on top makes the derivatives simple.
$\lim_{x\to 0^+} \dfrac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x)$	Differentiate top and bottom; $\frac{1/x}{-1/x^2} = -x$.
0	This is why entropy uses $0 \log 0 = 0$: $p \log p \to 0$ as $p \to 0^+$, so an outcome of probability 0 adds nothing.

Problem 5

Find $\lim_{x\to\infty} \dfrac{\ln x}{\sqrt{x}}$.

As $x \to \infty$: $\frac{\infty}{\infty}$.	Indeterminate.
$\lim_{x\to\infty} \dfrac{1/x}{1/(2\sqrt{x})} = \lim_{x\to\infty} \dfrac{2\sqrt{x}}{x}$	$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$.
$= \lim_{x\to\infty} \dfrac{2}{\sqrt{x}}$	$\frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$.
0	Any positive power of x eventually outgrows $\ln x$, even $x^{1/2}$.

Problem 6

Find $\lim_{x \to 0} \dfrac{1 - \cos x}{x^2}$.

At $x = 0$: $\frac{1-1}{0} = \frac{0}{0}$.	Indeterminate.
$\lim_{x \to 0} \dfrac{\sin x}{2x}$, still $\frac{0}{0}$	$(1 - \cos x)' = \sin x$.
$\lim_{x \to 0} \dfrac{\cos x}{2}$	A second application.
$\frac{1}{2}$	$\frac{\cos 0}{2}$. So $\cos x \approx 1 - \frac{x^2}{2}$ near 0.

Problem 7

Find $\lim_{x \to 0^+} x^x$.

As $x \to 0^+$: 0^0.	Indeterminate: a base going to 0 pulls toward 0, an exponent going to 0 pulls toward 1.
Let $y = x^x$; then $\ln y = x \ln x$.	Take the log to bring the exponent down.
$\lim_{x \to 0^+} \ln y = 0$	Problem 4.
1	$y = e^{\ln y}$, and e^x is continuous, so $y \to e^0 = 1$. The answer is not 0, which is the limit of the log (mistake 4).

Problem 8

Find $\lim_{x \to \infty} \left(1 + \dfrac{2}{x}\right)^x$.

As $x \to \infty$: 1^∞.	Indeterminate: the base goes to 1, but the exponent grows without bound.

$\ln y = x \ln\left(1 + \dfrac{2}{x}\right) = \dfrac{\ln(1 + 2/x)}{1/x}$, which has the form $\frac{0}{0}$.	Take the log, then write the product as a quotient.
$\lim\limits_{x \to \infty} \dfrac{\dfrac{1}{1 + 2/x} \cdot \left(-\dfrac{2}{x^2}\right)}{-\dfrac{1}{x^2}} = \lim\limits_{x \to \infty} \dfrac{2}{1 + 2/x} =$ 2	The chain rule on the top: $\left(\frac{2}{x}\right)' = -\frac{2}{x^2}$. The $-\frac{1}{x^2}$ cancels.
$e^2 \approx 7.389$	$y = e^{\ln y} \to e^2$. This is compound interest at a rate of 200% compounded ever more often.

Problem 9

Find $\lim\limits_{x \to 0} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$.

As $x \to 0^+$: $\infty - \infty$.	Indeterminate; from the left it is $-\infty + \infty$.
$\dfrac{1}{x} - \dfrac{1}{\sin x} = \dfrac{\sin x - x}{x \sin x}$, which has the form $\frac{0}{0}$.	Combine into one fraction over the common denominator.
$\lim\limits_{x \to 0} \dfrac{\cos x - 1}{\sin x + x \cos x}$, still $\frac{0}{0}$	The product rule on the bottom: $(x \sin x)' = \sin x + x \cos x$.
$\lim\limits_{x \to 0} \dfrac{-\sin x}{2 \cos x - x \sin x} = \dfrac{0}{2}$	Again: $(\sin x + x \cos x)' = \cos x + \cos x - x \sin x$.
0	From both sides. Near 0, $\sin x \approx x - \frac{x^3}{6}$, so $\frac{1}{\sin x} \approx \frac{1}{x}\left(1 + \frac{x^2}{6}\right)$ and the difference is about $-\frac{x}{6}$.

Problem 10

> Find $\lim_{x \to \infty} \dfrac{x + \sin x}{x}$. What happens if you use L'Hôpital's rule?

As $x \to \infty$: $\dfrac{\infty}{\infty}$.	Indeterminate.
The rule gives $\lim_{x \to \infty} \dfrac{1 + \cos x}{1}$, which does not exist.	$1 + \cos x$ keeps swinging between 0 and 2. The rule needs this limit to exist, so it gives no answer here, and it does not show that the original limit fails to exist.
$\dfrac{x + \sin x}{x} = 1 + \dfrac{\sin x}{x}$	Divide top and bottom by x instead.
$-\dfrac{1}{x} \le \dfrac{\sin x}{x} \le \dfrac{1}{x}$, so $\dfrac{\sin x}{x} \to 0$.	$\lvert \sin x \rvert \le 1$, and both bounds go to 0 (squeeze).
1; **L'Hôpital's rule does not apply, because $\lim(1 + \cos x)$ does not exist**	The rule is a sufficient condition, not a test: when the ratio of derivatives has no limit, use another method.

Where this goes wrong

1. Using the rule on a limit that is not indeterminate

The rule is used so often that it becomes the first step for every fraction.

$\lim_{x \to 1} \dfrac{x^2 + 1}{x + 1}$	A limit to find.
“It is a fraction, so differentiate top and bottom.”	The habit that causes the mistake: skipping the check of the form.
$\lim_{x \to 1} \dfrac{2x}{1} = 2$ ✗	At $x = 1$ the fraction is $\frac{2}{2} = 1$, not $\frac{0}{0}$ or $\frac{\infty}{\infty}$, so the rule does not apply, and the limit is 1 by substitution. Check the form every time, including before each repeat application.

2. The quotient rule instead of two separate derivatives

L'Hôpital's rule involves a fraction and derivatives, and the quotient rule is the derivative of a fraction.

$\lim_{x\to 0}\dfrac{\sin 3x}{x}$, form $\frac{0}{0}$	Right so far: Problem 1, step 1.
“Differentiate the fraction.”	The habit that causes the mistake: the rule remembered as "take the derivative".
$\lim_{x\to 0}\dfrac{3x\cos 3x-\sin 3x}{x^2}=0$ ✗	That is the limit of the derivative of $\frac{\sin 3x}{x}$, the slope of its graph near 0, which is a different question. The rule differentiates the top and the bottom separately: $\frac{3\cos 3x}{1}\to 3$.

3. Reading 1^∞ as 1

One to any power is one, so a base that goes to 1 seems to settle the limit.

$\lim_{x\to\infty}\left(1+\dfrac{2}{x}\right)^{x}$	Right so far: Problem 8.
“The base goes to 1, and $1^x=1$.”	The habit that causes the mistake: letting the base reach its limit while the exponent is still finite.
$\lim_{x\to\infty}\left(1+\dfrac{2}{x}\right)^{x}=1$ ✗	The base is never exactly 1, and the exponent grows as fast as the base approaches 1. At $x=100$ the value is already $1.02^{100}\approx 7.24$. Take the log: $x\ln\left(1+\frac{2}{x}\right)\to 2$, so the limit is $e^2\approx 7.389$.

4. Forgetting to undo the log

After taking a log, the limit that comes out looks like the end of the problem.

$y=x^x$, so $\ln y=x\ln x\to 0$ as $x\to 0^+$	Right so far: Problem 7, steps 2 and 3.

“The limit is 0.”	The habit that causes the mistake: reporting the limit of $\ln y$ as the limit of y.
$\lim_{x\to 0^+} x^x = 0$ ✗	0 is the limit of $\ln y$. The limit of y is $e^0 = 1$. A check: $0.01^{0.01} \approx 0.955$, near 1 and nowhere near 0.

5. Concluding the limit does not exist

When the rule produces something with no limit, it looks like the answer is "no limit".

$\lim_{x\to\infty} \dfrac{x+\sin x}{x}$, form $\frac{\infty}{\infty}$	Right so far: Problem 10.
“L'Hôpital: $\lim(1+\cos x)$ does not exist.”	Right so far as a fact about $1+\cos x$.
So $\lim_{x\to\infty} \dfrac{x+\sin x}{x}$ does not exist ✗	The rule only says that if the ratio of derivatives has a limit, the original ratio has the same one. It says nothing when the ratio of derivatives has no limit. Here $\frac{x+\sin x}{x} = 1 + \frac{\sin x}{x} \to 1$.

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