Law of sines and law of cosines

Ten triangles solved with the law of sines and the law of cosines: AAS, SAS and SSS, an obtuse angle found exactly, the ambiguous SSA case with two, one and no triangles, the area formula, a distance across a lake, and the angle between two vectors from their lengths, with worked solutions and the mistakes that lose the obtuse angle or the second triangle.

BEFORE YOU START

Two formulas solve every triangle that is not right-angled, and the law of cosines is the same identity as $\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\,u \cdot v$, the one behind cosine similarity. The work goes wrong in a few places: an arcsine that returns the acute angle when the triangle has an obtuse one, a second triangle missed when two sides and a non-included angle are given, a calculator in radian mode, and $\cos C$ applied to the wrong part of the formula. These ten problems cover each kind of given data, the ambiguous case in all three outcomes, area, and the vector form.

- Triangle ABC has angles A, B, C and sides a, b, c, with each side opposite the angle of the same letter. The angles sum to 180°. Angles on this page are in **degrees**, so a calculator must be in degree mode.
- Law of sines: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.
- Law of cosines: $c^2 = a^2 + b^2 - 2ab\cos C$, and the same with the letters rotated. Solved for the angle: $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$. When $C = 90^\circ$ it is Pythagoras.
- Which to use. Two angles and a side (AAS, ASA): law of sines. Two sides and the angle between them (SAS), or three sides (SSS): law of cosines. Two sides and an angle not between them (SSA): law of sines, and check for a second triangle.
- $\sin\theta = \sin(180^\circ - \theta)$, so an equation $\sin B = s$ with $0 < s < 1$ has two solutions between 0° and 180°, one acute and one obtuse. $\arcsin$ returns only the acute one. $\cos\theta$ is different: it is positive for acute angles and negative for obtuse ones, so $\arccos$ always returns the right angle of a triangle.
- Area $= \frac{1}{2}ab\sin C$: two sides and the angle between them.
- Exact values used: $\sin 30^\circ = \frac{1}{2}$, $\sin 45^\circ = \frac{\sqrt{2}}{2}$, $\sin 60^\circ = \frac{\sqrt{3}}{2}$, $\cos 60^\circ = \frac{1}{2}$, $\cos 120^\circ = -\frac{1}{2}$. Other values are rounded to two decimal places.

Builds on: Trig identities

Problems

1. In triangle ABC, $A = 45°$, $B = 60°$ and $a = 10$. Find C, b and c exactly. difficulty ·

2. In triangle ABC, $a = 5$, $b = 8$ and $C = 60°$. Find c. difficulty ·

3. A triangle has sides 7, 8 and 13. Find its largest angle exactly. difficulty ··

4. Solve the triangle with sides $a = 5$, $b = 6$, $c = 7$: find all three angles. difficulty ··

5. In triangle ABC, $A = 30°$, $a = 6$ and $b = 10$. Find every possible B, C and c. difficulty ··

6. In triangle ABC, $A = 30°$, $a = 12$ and $b = 10$. Find every possible B, C and c. difficulty ··

7. In triangle ABC, $A = 30°$, $a = 4$ and $b = 10$. Find B. difficulty ·

8. Find the area of the triangle with sides 5, 6 and 7 exactly. difficulty ··

9. From a point P on the shore, one end of a lake is 300 m away and the other end is 500 m away, and the angle at P between the two lines of sight is $120°$. How long is the lake? difficulty ···

10. Vectors u and v have $\|u\| = 3$, $\|v\| = 5$ and $\|u - v\| = 7$. Find $u \cdot v$ and the angle between u and v. difficulty ···

Answers

1. $C = 75^\circ$, $b = 5\sqrt{6} \approx 12.25$, $c = 5(\sqrt{3}+1) \approx 13.66$
2. $c = 7$
3. $C = 120^\circ$
4. $A \approx 44.42^\circ$, $B \approx 57.12^\circ$, $C \approx 78.46^\circ$
5. Two triangles: $B \approx 56.44^\circ$, $C \approx 93.56^\circ$, $c \approx 11.98$; or $B \approx 123.56^\circ$, $C \approx 26.44^\circ$, $c \approx 5.34$
6. One triangle: $B \approx 24.62^\circ$, $C \approx 125.38^\circ$, $c \approx 19.57$
7. No such triangle
8. $6\sqrt{6} \approx 14.70$
9. 700 m
10. $u \cdot v = -\frac{15}{2}$, $\theta = 120^\circ$

Worked solutions

Problem 1

> In triangle ABC, $A = 45^\circ$, $B = 60^\circ$ and $a = 10$. Find C, b and c exactly.

$C = 180^\circ - 45^\circ - 60^\circ = 75^\circ.$	Angle sum. Two angles and a side: law of sines.
$b = \dfrac{a \sin B}{\sin A} = \dfrac{10 \cdot \frac{\sqrt{3}}{2}}{\frac{\sqrt{2}}{2}} = \dfrac{10\sqrt{3}}{\sqrt{2}} = 5\sqrt{6}$	From $\frac{a}{\sin A} = \frac{b}{\sin B}$; then rationalize, $\frac{10\sqrt{3}}{\sqrt{2}} = \frac{10\sqrt{6}}{2}$.
$\sin 75^\circ = \sin(45^\circ + 30^\circ) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$	The sum formula $\sin(x+y) = \sin x \cos y + \cos x \sin y$, with $\cos 45^\circ = \frac{\sqrt{2}}{2}$ and $\cos 30^\circ = \frac{\sqrt{3}}{2}$.
$c = \dfrac{a \sin C}{\sin A} = \dfrac{10 \cdot \frac{\sqrt{6}+\sqrt{2}}{4}}{\frac{\sqrt{2}}{2}} = \dfrac{5(\sqrt{6}+\sqrt{2})}{\sqrt{2}} = 5(\sqrt{3}+1)$	Law of sines again; $\frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3}$.
$C = 75^\circ$, $b = 5\sqrt{6} \approx 12.25$, $c = 5(\sqrt{3}+1) \approx 13.66$	Sanity check: the largest angle, C, is opposite the largest side, c.

Problem 2

> In triangle ABC, $a = 5$, $b = 8$ and $C = 60^\circ$. Find c.

$c^2 = a^2 + b^2 - 2ab\cos C$	Two sides and the angle between them: law of cosines.
$c^2 = 25 + 64 - 2 \cdot 5 \cdot 8 \cdot \frac{1}{2}$	$\cos 60^\circ = \frac{1}{2}$. The $\cos C$ multiplies only the $2ab$ term.
$c^2 = 89 - 40 = 49$	Multiply before subtracting.
$c = 7$	A side length is positive, so take the positive root.

Problem 3

> A triangle has sides 7, 8 and 13. Find its largest angle exactly.

The largest angle is opposite the largest side, $c = 13$.	In any triangle, larger sides face larger angles.
$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab} = \dfrac{49 + 64 - 169}{2 \cdot 7 \cdot 8} = \dfrac{-56}{112}$	Law of cosines solved for the angle.
$\cos C = -\frac{1}{2}$	A negative cosine means C is obtuse: $a^2 + b^2 < c^2$.
$C = 120^\circ$	The angle between 0° and 180° whose cosine is $-\frac{1}{2}$. There is only one, which is why the law of cosines is safe here.

Problem 4

> Solve the triangle with sides $a = 5$, $b = 6$, $c = 7$: find all three angles.

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{36 + 49 - 25}{84} = \dfrac{60}{84} = \dfrac{5}{7}$, so $A \approx 44.42^\circ$.	Law of cosines for the angle opposite a.
$\cos B = \dfrac{a^2 + c^2 - b^2}{2ac} = \dfrac{25 + 49 - 36}{70} = \dfrac{38}{70}$, so $B \approx 57.12^\circ$.	The same formula with the letters rotated.
$C = 180^\circ - A - B \approx 78.46^\circ$.	Angle sum. Check with the law of cosines: $\cos C = \frac{25+36-49}{60} = 0.2$, and $\arccos 0.2 \approx 78.46^\circ$.
$A \approx 44.42^\circ$, $B \approx 57.12^\circ$, $C \approx 78.46^\circ$	All acute, since every cosine is positive; the largest is opposite the side 7.

Problem 5

> In triangle ABC, $A = 30°$, $a = 6$ and $b = 10$. Find every possible B, C and c.

$\sin B = \dfrac{b \sin A}{a} = \dfrac{10 \cdot \frac{1}{2}}{6} = \dfrac{5}{6}$	Two sides and a non-included angle (SSA): law of sines.
$B \approx 56.44°$ or $B \approx 180° - 56.44° = 123.56°$.	$\frac{5}{6} < 1$, so there are two angles with this sine, $\arcsin \frac{5}{6}$ and its supplement.
Both are possible: $30° + 123.56° = 153.56° < 180°$.	The obtuse B leaves room for a positive C, so it gives a real triangle too.
$C \approx 93.56°$, $c = \dfrac{a \sin C}{\sin A} = 12 \sin 93.56° \approx 11.98$; or $C \approx 26.44°$, $c = 12 \sin 26.44° \approx 5.34$.	Angle sum for each, then the law of sines with $\frac{a}{\sin A} = \frac{6}{1/2} = 12$.
Two triangles: $B \approx 56.44°$, $C \approx 93.56°$, $c \approx 11.98$; **or** $B \approx 123.56°$, $C \approx 26.44°$, $c \approx 5.34$	Geometrically: side $a = 6$ swings from C and meets the line of side c at two points, because 6 is between the height $b \sin A = 5$ and $b = 10$.

Problem 6

> In triangle ABC, $A = 30°$, $a = 12$ and $b = 10$. Find every possible B, C and c.

$\sin B = \dfrac{10 \cdot \frac{1}{2}}{12} = \dfrac{5}{12}$	SSA again: law of sines.
$B \approx 24.62°$ or $B \approx 155.38°$.	The two angles with sine $\frac{5}{12}$.
$30° + 155.38° > 180°$, so only $B \approx 24.62°$ is possible.	The obtuse candidate leaves no room for C. This always happens when $a \geq b$: the angle opposite the shorter side b must be smaller than A.

$C \approx 125.38^\circ$ and $c = \dfrac{12 \sin 125.38^\circ}{1/2} \approx 19.57$.	Angle sum, then the law of sines with $\frac{a}{\sin A} = 24$.
One triangle: $B \approx 24.62^\circ$, $C \approx 125.38^\circ$, $c \approx 19.57$	Steps 3 and 4.

Problem 7

In triangle ABC, $A = 30^\circ$, $a = 4$ and $b = 10$. Find B.

$\sin B = \dfrac{10 \cdot \frac{1}{2}}{4} = \dfrac{5}{4}$	Law of sines.
$\sin B \le 1$ for every angle, and $\frac{5}{4} > 1$.	No angle has this sine.
No such triangle	Geometrically: the height from C to the line of side c is $b \sin A = 5$, and a side of length 4 cannot reach that line.

Problem 8

Find the area of the triangle with sides 5, 6 and 7 exactly.

$\cos C = 0.2 = \frac{1}{5}$, from Problem 4.	The area formula needs an angle between two known sides; C is between $a = 5$ and $b = 6$.
$\sin C = \sqrt{1 - \frac{1}{25}} = \sqrt{\frac{24}{25}} = \dfrac{2\sqrt{6}}{5}$	$\sin^2 C + \cos^2 C = 1$, and $\sin C > 0$ for any angle of a triangle, so take the positive root.
$\text{Area} = \frac{1}{2} \cdot 5 \cdot 6 \cdot \dfrac{2\sqrt{6}}{5} = 6\sqrt{6}$	$\frac{1}{2}ab \sin C$.
$6\sqrt{6} \approx 14.70$	Check with Heron's formula: $s = 9$, $\sqrt{9 \cdot 4 \cdot 3 \cdot 2} = \sqrt{216} = 6\sqrt{6}$.

Problem 9

> From a point P on the shore, one end of a lake is 300 m away and the other end is 500 m away, and the angle at P between the two lines of sight is 120°. How long is the lake?

$d^2 = 300^2 + 500^2 - 2 \cdot 300 \cdot 500 \cos 120^\circ$	The two sight lines and the lake form a triangle with two sides and the angle between them: law of cosines.
$\cos 120^\circ = -\tfrac{1}{2}$, so $-2 \cdot 300 \cdot 500 \cdot \left(-\tfrac{1}{2}\right) = +150\,000.$	The angle is obtuse, so the cosine is negative and the last term is added.
$d^2 = 90\,000 + 250\,000 + 150\,000 = 490\,000$	The squares and the step 2 term.
700 **m**	$\sqrt{490\,000} = 700$. Sanity check: an angle over 90° makes the far side longer than Pythagoras would, $\sqrt{340\,000} \approx 583$.

Problem 10

> Vectors u and v have $\|u\| = 3$, $\|v\| = 5$ and $\|u - v\| = 7$. Find $u \cdot v$ and the angle between u and v.

$\|u - v\|^2 = (u - v) \cdot (u - v) = \|u\|^2 + \|v\|^2 - 2\,u \cdot v$	Expand the dot product: $u \cdot u - 2\,u \cdot v + v \cdot v$, using $u \cdot v = v \cdot u$.
$49 = 9 + 25 - 2\,u \cdot v$, so $u \cdot v = -\tfrac{15}{2}$.	Substitute and solve.
$\cos\theta = \dfrac{u \cdot v}{\|u\|\,\|v\|} = \dfrac{-15/2}{15} = -\tfrac{1}{2}$	The definition of the angle between two vectors. This ratio is the cosine similarity of u and v.
$u \cdot v = -\tfrac{15}{2}$, $\theta = 120^\circ$	Steps 1 and 3 together are the law of cosines in the triangle with sides $\|u\| = 3$, $\|v\| = 5$ and $\|u - v\| = 7$: its angle opposite the side 7 is 120°.

Where this goes wrong

1. Taking arcsin for an angle that may be obtuse

Once one angle is known, the law of sines looks like the quickest route to the next.

Sides 7, 8, 13: $\cos A = \frac{64+169-49}{2\cdot 8\cdot 13} = \frac{23}{26}$, so $\sin A = \frac{7\sqrt{3}}{26}$	Right so far: the angle opposite the side 7, from the law of cosines, $A \approx 27.80^\circ$.
$\sin C = \dfrac{13 \sin A}{7} = \dfrac{\sqrt{3}}{2}$	Right so far: the law of sines.
$C = \arcsin \frac{\sqrt{3}}{2} = 60^\circ$ ✗	arcsin returns only the acute angle, and $\sin 120^\circ$ is also $\frac{\sqrt{3}}{2}$. Here $C = 120^\circ$ (Problem 3). With $C = 60^\circ$, B would be $180^\circ - 27.80^\circ - 60^\circ = 92.20^\circ$, the largest angle, opposite $b = 8$, which is not the longest side. Use the law of cosines for the largest angle, or for any angle that might be obtuse.

2. Missing the second triangle in the SSA case

The law of sines gives one value of $\sin B$, and the calculator gives one angle for it.

$A = 30^\circ$, $a = 6$, $b = 10$: $\sin B = \frac{5}{6}$	Right so far: Problem 5, step 1.
"$\arcsin \frac{5}{6} \approx 56.44^\circ$, so $B \approx 56.44^\circ$."	The habit that causes the mistake: stopping at the calculator's answer.
The triangle has $B \approx 56.44^\circ$, $C \approx 93.56^\circ$, $c \approx 11.98$, and no other solution ✗	$180^\circ - 56.44^\circ = 123.56^\circ$ has the same sine and still leaves 26.44° for C, so a second triangle with $c \approx 5.34$ also fits the data. Whenever SSA gives $\sin B < 1$, test the supplement: it is a second triangle exactly when $A + (180^\circ - B) < 180^\circ$.

3. A calculator in radian mode

Nothing on the screen says which mode the calculator is in.

$A = 45°$, $B = 60°$, $a = 10$: $b = \dfrac{10 \sin 60°}{\sin 45°}$	Right so far: Problem 1, step 2.
"Type $10 \sin(60) / \sin(45)$."	The habit that causes the mistake: entering degrees into a calculator set to radians.
$b = \dfrac{10 \sin 60}{\sin 45} \approx -3.58$ ✗	In radians, $\sin 60 \approx -0.305$ and $\sin 45 \approx 0.851$: these are the sines of 60 and 45 radians. A negative side length is the giveaway. In degree mode, $b \approx 12.25$.

4. Multiplying the whole sum by cos C

The formula $c^2 = a^2 + b^2 - 2ab \cos C$ has $\cos C$ at the end, and it gets applied to everything before it.

$a = 5$, $b = 8$, $C = 60°$: $c^2 = 25 + 64 - 2 \cdot 5 \cdot 8 \cos 60°$	Right so far: Problem 2, step 2.
"$25 + 64 - 80 = 9$, then multiply by $\cos 60°$."	The habit that causes the mistake: working left to right instead of multiplying first.
$c^2 = (25 + 64 - 80) \cdot \frac{1}{2} = 4.5$, so $c \approx 2.12$ ✗	$\cos C$ multiplies only $2ab$: $c^2 = 89 - 80 \cdot \frac{1}{2} = 49$, so $c = 7$. A side of 2.12 cannot be right: by the triangle inequality, $c > b - a = 3$.

5. Dropping the sign of cos 120°

The law of cosines is remembered as "Pythagoras minus something".

$d^2 = 300^2 + 500^2 - 2 \cdot 300 \cdot 500 \cos 120°$	Right so far: Problem 9, step 1.
"$\cos 120°$ has reference angle $60°$, so it is $\frac{1}{2}$."	The habit that causes the mistake: the reference angle gives the size of the cosine, not its sign.

$d^2 = 340\,000 - 150\,000 = 190\,000$, so $d \approx 436$ m ✗	$\cos 120^\circ = -\frac{1}{2}$, since cosine is negative in quadrant II, so the term is added: $d^2 = 490\,000$, $d = 700$ m. A wide angle at P should give a long lake, longer than the Pythagoras value of about 583 m, not shorter.

More at eulerpad.app. This is one of 82 free practice sets with worked solutions at eulerpad.app/practice, alongside guides to the topics behind them.

EulerPad, the app, reads your handwritten math on iPad, iPhone and Android and checks it line by line. It's free to start.

eulerpad.app/practice/law-of-sines-and-law-of-cosines