Integration: substitution and parts

Ten integrals by substitution and by parts — including parts applied twice, a definite integral, an improper one, and the sigmoid whose antiderivative is softplus — with worked solutions and the mistakes that forget the du or pick u and dv the wrong way round.

BEFORE YOU START

Two techniques cover every integral on this page: substitution, which undoes the chain rule, and integration by parts, which undoes the product rule. The work goes wrong in a few places that recur: a dx replaced without the factor that du brings, u and dv chosen the wrong way round, a sign lost from du, limits left unchanged after a substitution, and a value at infinity asserted instead of computed. These ten problems run from single substitutions and single applications of parts to parts applied twice, a definite integral, an improper one, and the integral of the sigmoid.

- $\int f\,dx = F + C$ means $F' = f$: F is an antiderivative of f, and the constant C stands for any constant, since adding a constant does not change the derivative.
- Substitution: set $u = g(x)$, so $du = g'(x)\,dx$, and rewrite the whole integrand, dx included, in terms of u. In a definite integral, change the limits with it: $x = a$ becomes $u = g(a)$.
- Integration by parts: $\int u\,dv = uv - \int v\,du$. It is the product rule $(uv)' = u'v + uv'$ integrated and rearranged. For a definite integral, $\int_a^b u\,dv = \big[uv\big]_a^b - \int_a^b v\,du$, where $\big[F\big]_a^b = F(b) - F(a)$.
- u has a meaning local to each problem: in a substitution it is the new variable, in parts it is the factor to be differentiated, with dv the part to be integrated. No problem uses both at once.
- An improper integral is a limit: $\int_0^\infty f\,dx$ means $\lim_{b\to\infty} \int_0^b f\,dx$.
- $\ln$ is the natural log, defined for positive arguments; $\int \frac{du}{u} = \ln|u| + C$ on any interval where $u \neq 0$.
- Every antiderivative below can be checked by differentiating it — do that before looking at the solution.

Builds on: Differentiation rules: chain, product, quotient

Problems

1. $\int x e^{x^2}\,dx$ difficulty ·

2. $\int \frac{2x}{1+x^2}\,dx$ difficulty ·

3. $\int x e^{x}\,dx$ difficulty ··

4. $\int \ln x\,dx$ difficulty ··

5. $\int x \cos x\,dx$ difficulty ··

6. $\int e^{x} \sin x\,dx$ difficulty ···

7. $\int_0^1 \frac{dx}{1+x^2}$ difficulty ··

8. $\int \tan x\,dx$ difficulty ···

9. $\int_0^\infty x e^{-x}\,dx$ difficulty ···

10. $\int \sigma(x)\,dx$ where $\sigma(x) = \frac{1}{1+e^{-x}}$, and then $\int_{-\infty}^{0} \sigma(x)\,dx$. difficulty ···

Answers

1. $\frac{1}{2}e^{x^2} + C$
2. $\ln(1 + x^2) + C$
3. $(x - 1)e^x + C$
4. $x \ln x - x + C$
5. $x \sin x + \cos x + C$
6. $\frac{1}{2}e^x(\sin x - \cos x) + C$
7. $\frac{\pi}{4}$
8. $-\ln|\cos x| + C$
9. 1
10. $\ln(1 + e^x) + C$; and $\int_{-\infty}^{0} \sigma = \ln 2 - \lim_{a \to -\infty} \ln(1 + e^a) = \ln 2$

Worked solutions

Problem 1

$$\int x e^{x^2}\,dx$$

$u = x^2,\ du = 2x\,dx.$	e^{x^2} is e^u with inner function x^2, and the integrand also contains x, which is half of that inner function's derivative $2x$.
$x\,dx = \tfrac{1}{2}\,du.$	Divide both sides of step 1 by 2, so that the $x\,dx$ in the integral can be replaced as a whole.
$\displaystyle\int x e^{x^2}\,dx = \int e^u \cdot \tfrac{1}{2}\,du = \tfrac{1}{2}e^u + C.$	Substitute steps 1 and 2; $(e^u)' = e^u$, and the constant $\tfrac{1}{2}$ comes out of the integral.
$\tfrac{1}{2}e^{x^2} + C$	Put $u = x^2$ back. Check: by the chain rule, $\tfrac{d}{dx}\tfrac{1}{2}e^{x^2} = \tfrac{1}{2}e^{x^2}\cdot 2x = x e^{x^2}$.

Problem 2

$$\int \frac{2x}{1+x^2}\,dx$$

$u = 1 + x^2,\ du = 2x\,dx.$	The numerator $2x\,dx$ is exactly the derivative of the denominator, times dx.
$\displaystyle\int \frac{2x}{1+x^2}\,dx = \int \frac{du}{u} = \ln\lvert u\rvert + C.$	The denominator becomes u and the numerator with dx becomes du.
$\ln(1+x^2) + C$	Put $u = 1 + x^2$ back; $1 + x^2 \geq 1 > 0$ for every x, so the absolute value can be dropped and the answer holds on the whole real line.

Problem 3

> $\displaystyle\int x e^{x}\,dx$

$u = x,\ dv = e^{x}\,dx;\ du = dx,\ v = e^{x}.$	Differentiating x makes it simpler, the constant 1, and e^{x} is its own antiderivative, so the new integral $\int v\,du$ has no factor of x left.
$\displaystyle\int x e^{x}\,dx = x e^{x} - \int e^{x}\,dx.$	Parts: $uv - \int v\,du$.
$x e^{x} - e^{x} + C.$	$\int e^{x}\,dx = e^{x} + C$.
$(x-1)e^{x} + C$	Factor out e^{x}. Check: by the product rule, $\big((x-1)e^{x}\big)' = e^{x} + (x-1)e^{x} = x e^{x}$.

Problem 4

> $\displaystyle\int \ln x\,dx$

$u = \ln x,\ dv = dx;\ du = \dfrac{dx}{x},\ v = x.$	No antiderivative of $\ln x$ is known yet, but its derivative, $1/x$, is simple; the other factor is then 1, and dx integrates to x. $\ln x$ needs $x > 0$.
$\displaystyle\int \ln x\,dx = x\ln x - \int x \cdot \frac{1}{x}\,dx.$	Parts: $uv - \int v\,du$.
$x\ln x - \int 1\,dx = x\ln x - x + C.$	$x \cdot \frac{1}{x} = 1$ for $x > 0$, and $\int 1\,dx = x + C$.
$x\ln x - x + C$	Check: $(x\ln x - x)' = \ln x + x\cdot\frac{1}{x} - 1 = \ln x$.

Problem 5

> $\int x \cos x \, dx$

$u = x$, $dv = \cos x \, dx$; $du = dx$, $v = \sin x$.	Differentiating x removes it, and $\cos x$ integrates to $\sin x$ without getting harder.
$\int x \cos x \, dx = x \sin x - \int \sin x \, dx$.	Parts: $uv - \int v \, du$.
$\int \sin x \, dx = -\cos x + C$.	$(-\cos x)' = \sin x$.
$x \sin x + \cos x + C$	Subtracting $-\cos x$ adds $\cos x$. Check: $(x \sin x + \cos x)' = \sin x + x \cos x - \sin x = x \cos x$.

Problem 6

> $\int e^x \sin x \, dx$

Call the integral $I = \int e^x \sin x \, dx$.	Neither factor gets simpler when differentiated, so parts will not finish the job; the plan is to apply parts twice until I reappears, then solve for it.
$u = \sin x$, $dv = e^x \, dx$; $du = \cos x \, dx$, $v = e^x$.	e^x integrates to itself, so v stays as simple as dv.
$I = e^x \sin x - \int e^x \cos x \, dx$.	Parts: $uv - \int v \, du$.
For $\int e^x \cos x \, dx$: $u = \cos x$, $dv = e^x \, dx$; $du = -\sin x \, dx$, $v = e^x$.	Keep the same choice, trig as u and the exponential as dv. Swapping it would integrate the $\cos x$ back to $\sin x$ and undo step 3, giving only $I = I$.

$\int e^x \cos x \, dx = e^x \cos x - \int e^x (-\sin x) \, dx = e^x \cos x + I.$	Parts again; the minus from du turns the new integral into $+I$.
$I = e^x \sin x - e^x \cos x - I.$	Substitute step 5 into step 3. The original integral now appears on both sides.
$2I = e^x (\sin x - \cos x).$	Add I to both sides and factor out e^x. The constants of integration from the two applications of parts are gathered into one C in the final line.
$\frac{1}{2} e^x (\sin x - \cos x) + C$	Divide by 2. Check: $\left(\frac{1}{2} e^x (\sin x - \cos x)\right)' = \frac{1}{2} e^x (\sin x - \cos x) + \frac{1}{2} e^x (\cos x + \sin x) = e^x \sin x.$

Problem 7

$$\int_0^1 \frac{dx}{1 + x^2}$$

$y = \arctan x$ means $\tan y = x$ with $-\frac{\pi}{2} < y < \frac{\pi}{2}$.	arctan is the inverse of tan on $(-\frac{\pi}{2}, \frac{\pi}{2})$, where $\cos y > 0$ and tan is increasing, so each x has exactly one y.
$\frac{1}{\cos^2 y} \, y' = 1.$	Differentiate $\tan y = x$ with respect to x: the chain rule gives the derivative of tan at y, which is $1/\cos^2 y$ (quotient rule on $\sin y / \cos y$), times y'.
$y' = \cos^2 y = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}.$	$1 + \tan^2 y = \frac{\cos^2 y + \sin^2 y}{\cos^2 y} = \frac{1}{\cos^2 y}$, and $\tan y = x$.
$\int_0^1 \frac{dx}{1 + x^2} = \left[\arctan x\right]_0^1 = \arctan 1 - \arctan 0.$	Step 3 makes $\arctan x$ an antiderivative of the integrand, and a definite integral is the antiderivative's value at the upper limit minus its value at the lower one.
$\arctan 1 = \frac{\pi}{4}$ and $\arctan 0 = 0.$	$\tan \frac{\pi}{4} = 1$ and $\tan 0 = 0$, and both angles lie in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

$\dfrac{\pi}{4}$	$\frac{\pi}{4} - 0$. The integrand lies between $\frac{1}{2}$ and 1 on $[0, 1]$, and $\frac{\pi}{4} \approx 0.785$ is between them.

Problem 8

> $\displaystyle\int \tan x \, dx$

$\displaystyle\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$, on an interval where $\cos x \neq 0$.	$\tan x$ is defined only where $\cos x \neq 0$; written as a quotient, the numerator is, up to sign, the derivative of the denominator.
$u = \cos x$, $du = -\sin x \, dx$, so $\sin x \, dx = -du$.	$(\cos x)' = -\sin x$; the minus sign has to be carried into the integral.
$\displaystyle\int \frac{\sin x}{\cos x} \, dx = \int \frac{-du}{u} = -\ln \lvert u \rvert + C.$	$\int du/u = \ln \lvert u \rvert + C$ for $u \neq 0$; the absolute value is needed because $\cos x$ is negative on half of the intervals where $\tan x$ is defined.
$-\ln \lvert \cos x \rvert + C$	Put $u = \cos x$ back. Check: where $\cos x > 0$, the chain rule gives $\frac{d}{dx} \ln \cos x = \dfrac{-\sin x}{\cos x} = -\tan x$; where $\cos x < 0$, it gives $\frac{d}{dx} \ln(-\cos x) = \dfrac{\sin x}{-\cos x} = -\tan x$. Either way the minus sign in front makes the derivative $\tan x$.

Problem 9

> $\displaystyle\int_0^\infty x e^{-x} \, dx$

$\displaystyle\int_0^\infty x e^{-x} \, dx = \lim_{b \to \infty} \int_0^b x e^{-x} \, dx.$	The upper limit is not a number, so the integral is defined as the limit of integrals over $[0, b]$.
$u = x$, $dv = e^{-x} \, dx$; $du = dx$, $v = -e^{-x}$.	Differentiating x removes it; $(-e^{-x})' = e^{-x}$ by the chain rule.

$\displaystyle\int_0^b xe^{-x}\,dx = \big[-xe^{-x}\big]_0^b + \int_0^b e^{-x}\,dx.$	Parts on $[0,b]$: $\big[uv\big]_0^b - \int_0^b v\,du$, and $-v = e^{-x}$.
$\big[-xe^{-x}\big]_0^b = -be^{-b} - 0$ and $\displaystyle\int_0^b e^{-x}\,dx = \big[-e^{-x}\big]_0^b = 1 - e^{-b}$.	Evaluate each bracket at b and at 0; at $x = 0$ the first bracket is $0 \cdot e^0 = 0$.
$\displaystyle\int_0^b xe^{-x}\,dx = 1 - e^{-b} - be^{-b}.$	Add the two parts of step 4.
$0 < be^{-b} = \dfrac{b}{e^b} < \dfrac{b}{b^2/2} = \dfrac{2}{b}$ for $b > 0$, so $be^{-b} \to 0$.	$e^b = 1 + b + \frac{b^2}{2} + \cdots > \frac{b^2}{2}$ for $b > 0$, since every term of the series is positive; the bound $2/b$ goes to 0.
1	Take $b \to \infty$ in step 5: $e^{-b} \to 0$ and, by step 6, $be^{-b} \to 0$. This is the mean of the exponential distribution with rate 1.

Problem 10

> $\displaystyle\int \sigma(x)\,dx$ where $\sigma(x) = \frac{1}{1+e^{-x}}$, and then $\displaystyle\int_{-\infty}^{0} \sigma(x)\,dx$.

$\sigma(x) = \dfrac{e^x}{1+e^x}.$	Multiply top and bottom by e^x, which is never 0, so that the numerator becomes the derivative of the denominator.
$u = 1 + e^x$, $du = e^x\,dx$.	The denominator's derivative, e^x, is the numerator.
$\displaystyle\int \frac{e^x}{1+e^x}\,dx = \int \frac{du}{u} = \ln\lvert u\rvert + C.$	Substitute step 2.
$\displaystyle\int \sigma(x)\,dx = \ln(1+e^x) + C.$	Put $u = 1 + e^x$ back; $1 + e^x > 1$, so the absolute value can be dropped. This is softplus, whose derivative is σ (Differentiation rules, Problem 4).

$\displaystyle\int_{-\infty}^{0} \sigma(x)\,dx = \lim_{a\to-\infty}\int_{a}^{0}\sigma(x)\,dx = \lim_{a\to-\infty}\big(\ln 2 - \ln(1+e^{a})\big).$	The lower limit is not a number, so the integral is a limit; step 4 evaluated at 0 gives $\ln 2$, since $1+e^{0}=2$, and at a gives $\ln(1+e^{a})$.
$\ln(1+e^{a}) \to \ln 1 = 0$ as $a \to -\infty$.	$e^{a}\to 0$, and $\ln$ is continuous at 1.
$\ln(1+e^{x})+C$; **and** $\displaystyle\int_{-\infty}^{0}\sigma = \ln 2 - \lim_{a\to-\infty}\ln(1+e^{a}) = \ln 2$	Steps 4 to 6. The area is finite although σ never reaches 0, because $\sigma(x) < e^{x}$ decays exponentially as $x\to-\infty$.

Where this goes wrong

1. Substituting without changing dx

Setting $u = x^2$ turns e^{x^2} into e^{u} at a glance, and the rest of the integrand is easy to wave through.

$\displaystyle\int xe^{x^2}\,dx$ with $u = x^2$	Right so far: the substitution of Problem 1.
"e^{x^2} becomes e^{u}, and $x\,dx$ becomes du."	The shortcut that causes the mistake: $x\,dx$ treated as the whole of du.
$\displaystyle\int xe^{x^2}\,dx = \int e^{u}\,du = e^{x^2} + C$ ✗	$du = 2x\,dx$, so $x\,dx = \frac{1}{2}\,du$, and the factor $\frac{1}{2}$ is lost. Differentiating the answer shows it: $(e^{x^2})' = 2xe^{x^2}$, twice the integrand. The correct antiderivative is $\frac{1}{2}e^{x^2}+C$.

2. Parts with u and dv swapped

Parts works with either choice of u and dv in the sense that the formula stays true, so it is tempting to take the first factor that integrates easily.

$\displaystyle\int xe^{x}\,dx$ by parts	Right so far: a product of a polynomial and an exponential, the case parts is for.
"x integrates easily too, so either factor can be dv."	The reasoning that causes the mistake: choosing dv by what can be integrated, without asking what the new integral will look like.

$u = e^x$, $dv = x\,dx$ in $\displaystyle\int xe^x\,dx$ ✗	This gives $v = \frac{x^2}{2}$, $du = e^x\,dx$ and $\displaystyle\int xe^x\,dx = \frac{x^2}{2}e^x - \int \frac{x^2}{2}e^x\,dx$ — true, but the new integral is harder than the old one, which is the sign that the choice is wrong. Pick u to be what gets simpler when differentiated: $u = x$ gives $(x - 1)e^x + C$ (Problem 3).

3. Sign error in ∫ tan

$\tan x = \sin x / \cos x$ looks like a derivative over a function, and $\int u'/u = \ln|u|$ is the pattern that comes to mind.

$\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$ with $u = \cos x$	Right so far: the substitution of Problem 8.
"The numerator is the derivative of the denominator, so the answer is ln of the denominator."	The pattern that causes the mistake: $\sin x$ taken as the derivative of $\cos x$.
$\displaystyle\int \tan x\,dx = \ln\lvert\cos x\rvert + C$ ✗	$du = -\sin x\,dx$ brings a minus: $\sin x\,dx = -du$, so the integral is $-\ln\lvert\cos x\rvert + C$. Differentiating this line shows it: $(\ln\lvert\cos x\rvert)' = -\tan x$.

4. Changing the variable but keeping the old limits

A substitution in an indefinite integral changes only the integrand, and the habit carries over to definite integrals.

$\displaystyle\int_0^1 \frac{2x}{1+x^2}\,dx$ with $u = 1 + x^2$, $du = 2x\,dx$	Right so far: the substitution of Problem 2.
"The integrand becomes du/u; the rest stays."	The habit that causes the mistake: the limits, which are values of x, left in place.

$\displaystyle\int_0^1 \frac{2x}{1+x^2}\,dx = \int_0^1 \frac{du}{u}$ ✗	The limits must be values of u: $u(0) = 1$ and $u(1) = 2$, so the integral is $\int_1^2 du/u = \ln 2$. With the old limits the integral diverges, because $1/u$ is not integrable near $u = 0$ — yet the original integrand is continuous and bounded on $[0, 1]$, so its integral is finite.

5. Evaluating ∞ · 0 by feel

In Problem 9 the boundary term at infinity is x, which grows, times e^{-x}, which shrinks, and it is tempting to call the product 0 and move on.

$\displaystyle\int_0^\infty xe^{-x}\,dx = \big[-xe^{-x}\big]_0^\infty + \int_0^\infty e^{-x}\,dx$	Right so far as a plan: parts with $u = x$, $dv = e^{-x}\,dx$, as in Problem 9.
"At infinity e^{-x} is 0, so $-xe^{-x}$ is 0 there."	The reasoning that causes the mistake: $\infty \cdot 0$ read as 0.
$\big[-xe^{-x}\big]_0^\infty = 0 - 0$ ✗	The value happens to be right here, because $be^{-b} \to 0$ (Problem 9, step 6), but the line asserts it without a limit, and $\infty \cdot 0$ is not a value: $b \cdot \frac{1}{b} \to 1$, while $b \cdot b^{-1/2} \to \infty$. The same shortcut on $\int_1^\infty x \cdot \frac{1}{2}x^{-3/2}\,dx$, by parts with $u = x$ and $v = -x^{-1/2}$, calls the boundary term $\big[-x \cdot x^{-1/2}\big]_1^\infty$ zero at infinity, but at b it is $-\sqrt{b} \to -\infty$, and that integral diverges. Writing $\lim_{b\to\infty}$ is where the answer and the convergence are decided.

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