Improper integrals: infinite limits, infinite integrands and comparison

Ten improper integrals: 1/x³ to infinity, the p-test at infinity and at zero, an integrand that blows up inside the interval, ln x on (0, 1], 1/(1 + x²) over the whole line, x e^(−x²), a comparison that proves convergence, a partial-fraction limit that gives ln 2, and the Cauchy distribution whose mean does not exist, with worked solutions and the mistakes of integrating straight through a singularity, canceling ∞ − ∞ and comparing in the wrong direction.

BEFORE YOU START

Probability lives on improper integrals: a density has to integrate to 1 over an infinite range, and a mean or a variance exists only if another improper integral converges. The definition is always the same: replace the infinite limit, or the point where the integrand blows up, with a variable, integrate, and take a limit. When no antiderivative is available, a comparison with $\frac{1}{x^p}$ often settles convergence. The work goes wrong in a few places: a singularity inside the interval integrated straight through, two divergent pieces subtracted as if $\infty - \infty = 0$, a symmetric limit taken as the value of a two-sided integral, a comparison made in the direction that proves nothing, and the p-test used with the wrong threshold near 0. These ten problems cover each of them.

- **Infinite limit.** $\displaystyle\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^{b} f(x)\,dx$. It **converges** if the limit is a finite number and **diverges** otherwise.
- **Infinite integrand.** If f blows up at a, then $\displaystyle\int_a^{b} f(x)\,dx = \lim_{\varepsilon\to 0^+}\int_{a+\varepsilon}^{b} f(x)\,dx$. If it blows up at a point c inside $[a, b]$, split at c; the integral converges only if both pieces do.
- **Both ends infinite.** $\displaystyle\int_{-\infty}^{\infty} f = \int_{-\infty}^{0} f + \int_{0}^{\infty} f$, with each piece a separate limit. Both must converge.
- **The p-test.** $\displaystyle\int_1^{\infty} \frac{dx}{x^p}$ converges exactly when $p > 1$, and $\displaystyle\int_0^{1} \frac{dx}{x^p}$ converges exactly when $p < 1$. At $p = 1$ both diverge.
- **Comparison.** If $0 \le f(x) \le g(x)$ for $x \ge a$: if $\int_a^{\infty} g$ converges, so does $\int_a^{\infty} f$; if $\int_a^{\infty} f$ diverges, so does $\int_a^{\infty} g$. Being smaller than something divergent, or larger than something convergent, proves nothing.

Builds on: Integration: substitution and parts, Partial fractions and integrating rational functions, L'Hôpital's rule and indeterminate forms

Problems

1. Evaluate $\displaystyle\int_1^\infty \frac{dx}{x^3}$. difficulty ·

2. Show that $\displaystyle\int_1^\infty \frac{dx}{x^p}$ converges exactly when $p > 1$, and find its value. Use the result on $p = \frac{3}{2}$ and $p = 1$. difficulty ··

3. Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$, then say for which p the integral $\displaystyle\int_0^1 \frac{dx}{x^p}$ converges. difficulty ··

4. Does $\displaystyle\int_{-1}^2 \frac{dx}{x^2}$ converge? difficulty ··

5. Evaluate $\displaystyle\int_0^1 \ln x \, dx$. difficulty ··

6. Evaluate $\displaystyle\int_{-\infty}^\infty \frac{dx}{1+x^2}$. difficulty ··

7. Evaluate $\displaystyle\int_0^\infty x e^{-x^2}\, dx$. difficulty ··

8. Show that $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^3+1}}$ converges, and give an upper bound for it. difficulty ···

9. Evaluate $\displaystyle\int_1^\infty \frac{dx}{x(x+1)}$. difficulty ···

10. Does $\displaystyle\int_{-\infty}^\infty \frac{x}{1+x^2}\, dx$ converge? What does the answer say about the mean of the Cauchy distribution, whose density is $\displaystyle\frac{1}{\pi(1+x^2)}$? difficulty ···

Answers

1. $\frac{1}{2}$
2. Converges to $\frac{1}{p-1}$ for $p > 1$, diverges for $p \le 1$; $p = \frac{3}{2}$ gives 2, $p = 1$ diverges
3. 2; $\int_0^1 x^{-p}\,dx$ converges to $\frac{1}{1-p}$ exactly when $p < 1$
4. No: the integral diverges
5. -1
6. π
7. $\frac{1}{2}$
8. It converges, and its value is less than 2
9. $\ln 2 \approx 0.693$
10. It diverges, so the Cauchy distribution has no mean

Worked solutions

Problem 1

> Evaluate $\displaystyle\int_1^{\infty} \frac{dx}{x^3}$.

$\displaystyle\int_1^{b} x^{-3}\,dx = \Big[-\frac{1}{2x^2}\Big]_1^{b} = \frac{1}{2} - \frac{1}{2b^2}$	Replace ∞ with b and integrate.
$\displaystyle\frac{1}{2b^2} \to 0$ as $b \to \infty$	Take the limit.
$\frac{1}{2}$	The integral converges: the area under $\frac{1}{x^3}$ from 1 out to infinity is finite.

Problem 2

> Show that $\displaystyle\int_1^{\infty} \frac{dx}{x^p}$ converges exactly when $p > 1$, and find its value. Use the result on $p = \frac{3}{2}$ and $p = 1$.

For $p \neq 1$: $\displaystyle\int_1^{b} x^{-p}\,dx = \frac{b^{1-p} - 1}{1-p}$	The power rule.
If $p > 1$, $b^{1-p} \to 0$ and the integral is $\displaystyle\frac{-1}{1-p} = \frac{1}{p-1}$; if $p < 1$, $b^{1-p} \to \infty$	The exponent $1 - p$ is negative in the first case and positive in the second.
For $p = 1$: $\displaystyle\int_1^{b} \frac{dx}{x} = \ln b \to \infty$	The borderline case diverges, slowly.
Converges to $\frac{1}{p-1}$ for $p > 1$, diverges for $p \leq 1$; $p = \frac{3}{2}$ gives 2, $p = 1$ diverges	$\frac{1}{x}$ and $\frac{1}{x^{1.01}}$ look almost the same, but only the second has a finite area.

Problem 3

> Evaluate $\int_0^1 \frac{dx}{\sqrt{x}}$, then say for which p the integral $\int_0^1 \frac{dx}{x^p}$ converges.

$\frac{1}{\sqrt{x}}$ blows up at $x = 0$, so $\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0^+} \left[2\sqrt{x}\right]_\varepsilon^1$	The problem is at the left end.
$= \lim_{\varepsilon \to 0^+} \left(2 - 2\sqrt{\varepsilon}\right) = 2$	Converges.
For $p \neq 1$: $\int_\varepsilon^1 x^{-p}\,dx = \frac{1 - \varepsilon^{1-p}}{1-p}$	The same power rule as Problem 2.
$\varepsilon^{1-p} \to 0$ when $p < 1$ and $\to \infty$ when $p > 1$; $p = 1$ gives $-\ln \varepsilon \to \infty$	Near 0 the roles of the exponents are reversed.
2; $\int_0^1 x^{-p}\,dx$ **converges to** $\frac{1}{1-p}$ **exactly when** $p < 1$	Near 0 a small power blows up slowly enough, and near ∞ a large power decays fast enough. $p = 1$ fails at both ends.

Problem 4

> Does $\int_{-1}^2 \frac{dx}{x^2}$ converge?

$\frac{1}{x^2}$ blows up at $x = 0$, inside $[-1, 2]$	Check the interval for a zero in a denominator before integrating.
$\int_{-1}^2 = \int_{-1}^0 + \int_0^2$, and both pieces must converge	Split at the singularity.
$\int_\varepsilon^2 \frac{dx}{x^2} = \frac{1}{\varepsilon} - \frac{1}{2} \to \infty$	The right piece diverges: this is $p = 2 \geq 1$ near 0.
No: the integral diverges	One divergent piece is enough. The left piece diverges as well.

Problem 5

> Evaluate $\displaystyle\int_0^1 \ln x \, dx$.

$\ln x \to -\infty$ as $x \to 0^+$, so take $\displaystyle\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \ln x \, dx$	An infinite integrand at the left end.
$\displaystyle\int \ln x \, dx = x \ln x - x$	By parts with $u = \ln x$, $dv = dx$.
$\displaystyle\int_\varepsilon^1 \ln x \, dx = (0 - 1) - (\varepsilon \ln \varepsilon - \varepsilon)$	Evaluate at the limits.
$\varepsilon \ln \varepsilon = \dfrac{\ln \varepsilon}{1/\varepsilon} \to 0$	L'Hôpital: $\dfrac{1/\varepsilon}{-1/\varepsilon^2} = -\varepsilon \to 0$. The same limit justifies $0 \log 0 = 0$ in entropy.
-1	Negative, because $\ln x < 0$ on $(0, 1)$: the region lies below the axis.

Problem 6

> Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}$.

$\displaystyle\int_{-\infty}^{\infty} = \int_{-\infty}^{0} + \int_{0}^{\infty}$	Both ends are infinite, so split and take two separate limits.
$\displaystyle\int_0^b \frac{dx}{1 + x^2} = \arctan b \to \frac{\pi}{2}$	arctan is the antiderivative, and it levels off at $\frac{\pi}{2}$.
$\displaystyle\int_a^0 \frac{dx}{1 + x^2} = -\arctan a \to \frac{\pi}{2}$ as $a \to -\infty$	The integrand is even, so the left half matches the right.
π	Both pieces converge. Dividing by π gives the Cauchy density $\frac{1}{\pi(1+x^2)}$, which integrates to 1 (see Problem 10).

Problem 7

> Evaluate $\displaystyle\int_0^{\infty} xe^{-x^2}\,dx$.

Let $u = x^2$, $du = 2x\,dx$	The x in front is exactly what the substitution needs.
$\displaystyle\int_0^{b} xe^{-x^2}\,dx = \frac{1}{2}\int_0^{b^2} e^{-u}\,du = \frac{1}{2}\left(1 - e^{-b^2}\right)$	Change the limits with the variable: u runs from 0 to b^2.
$e^{-b^2} \to 0$	Take the limit.
$\frac{1}{2}$	The Gaussian e^{-x^2} on its own has no elementary antiderivative; with the extra x it does.

Problem 8

> Show that $\displaystyle\int_1^{\infty} \frac{dx}{\sqrt{x^3+1}}$ converges, and give an upper bound for it.

$x^3 + 1 > x^3$, so $\sqrt{x^3+1} > x^{3/2}$ and $\dfrac{1}{\sqrt{x^3+1}} < \dfrac{1}{x^{3/2}}$ for $x \geq 1$	A bigger denominator makes a smaller fraction.
$\displaystyle\int_1^{\infty} \frac{dx}{x^{3/2}} = \frac{1}{3/2 - 1} = 2$	The p-test with $p = \frac{3}{2} > 1$ (Problem 2).
The integrand is positive and below a function whose integral converges	Comparison in the direction that proves convergence.
It converges, and its value is less than 2	Numerically it is about 1.895. There is no elementary antiderivative, so the comparison is the proof.

Problem 9

> Evaluate $\displaystyle\int_1^\infty \frac{dx}{x(x+1)}$.

$\displaystyle\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$	Partial fractions, by the cover-up rule.
$\displaystyle\int_1^b \left(\frac{1}{x} - \frac{1}{x+1}\right) dx = \left[\ln \frac{x}{x+1}\right]_1^b = \ln \frac{b}{b+1} - \ln \frac{1}{2}$	Combine the logs before taking any limit.
$\displaystyle\frac{b}{b+1} \to 1$, so $\displaystyle\ln \frac{b}{b+1} \to 0$	The two logs grow together and their difference settles down.
$\ln 2 \approx 0.693$	The integrand behaves like $\frac{1}{x^2}$ for large x, so convergence was expected; the split pieces $\frac{1}{x}$ and $\frac{1}{x+1}$ each diverge on their own.

Problem 10

> Does $\displaystyle\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$ converge? What does the answer say about the mean of the Cauchy distribution, whose density is $\displaystyle\frac{1}{\pi(1+x^2)}$?

Split at 0: $\displaystyle\int_0^b \frac{x}{1+x^2}\,dx = \frac{1}{2}\ln(1+b^2)$	Substitute $u = 1 + x^2$.
$\frac{1}{2}\ln(1+b^2) \to \infty$ as $b \to \infty$	The right half diverges, and by symmetry the left half goes to $-\infty$.
The integral diverges	A two-sided integral needs both halves to converge on their own.
A symmetric cutoff gives $\displaystyle\int_{-b}^{b} \frac{x}{1+x^2}\,dx = 0$, but $\displaystyle\int_{-b}^{2b} \frac{x}{1+x^2}\,dx = \frac{1}{2}\ln\frac{1+4b^2}{1+b^2} \to \ln 2$	Different ways of letting the limits go to infinity give different answers, which is why neither one is the value.

It diverges, so the Cauchy distribution has no mean	The mean would be $\frac{1}{\pi}\int x \cdot \frac{1}{1+x^2}\,dx$, this integral divided by π. Sample averages of Cauchy draws never settle down.

Where this goes wrong

1. Integrating straight through a singularity

The antiderivative exists on both sides of the problem point, so the formula $F(b) - F(a)$ seems to apply.

$\int_{-1}^{2} \frac{dx}{x^2}$, with antiderivative $-\frac{1}{x}$	Right so far: Problem 4.
"Evaluate the antiderivative at the limits."	The habit that causes the mistake: the fundamental theorem used without checking that the integrand is continuous on $[-1, 2]$.
$\left[-\frac{1}{x}\right]_{-1}^{2} = -\frac{1}{2} - 1 = -\frac{3}{2}$ ✗	A positive integrand cannot have a negative integral. $\frac{1}{x^2}$ blows up at 0, inside the interval, and the integral diverges: $\int_{\varepsilon}^{2} \frac{dx}{x^2} = \frac{1}{\varepsilon} - \frac{1}{2}$, which is 99.5 at $\varepsilon = 0.01$ and keeps growing.

2. Canceling ∞ − ∞

After partial fractions the integral looks like two simpler ones.

$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$	Right so far: Problem 9, step 1.
"Integrate each term from 1 to ∞ separately."	The habit that causes the mistake: splitting an improper integral into pieces that are themselves improper.
$\int_1^\infty \frac{dx}{x} - \int_1^\infty \frac{dx}{x+1} = \infty - \infty$, so the integral diverges ✗	$\infty - \infty$ has no value, so it decides nothing. Keep the pieces together up to a finite b: $\ln\frac{b}{b+1} + \ln 2 \to \ln 2 \approx 0.693$. The integral converges.

3. Taking a symmetric limit as the value

For an odd integrand, cutting off at $-b$ and b gives 0 every time, which looks like the answer.

$\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$, and $\int_{-b}^{b} \frac{x}{1+x^2}\,dx = 0$ for every b	Right so far: Problem 10, step 4.
"So the limit as $b \to \infty$ is 0."	The habit that causes the mistake: one particular way of going to infinity treated as the definition.
The integral is 0, so the Cauchy distribution has mean 0 ✗	The definition takes the two ends separately, and $\int_0^b$ already goes to ∞. Cutting off at $-b$ and $2b$ gives $\ln 2 \approx 0.693$ instead of 0. The integral diverges, and the Cauchy distribution has no mean at all, though its median is 0.

4. Comparing in the wrong direction

Comparison only works one way in each case, and the wrong way still looks like an argument.

$\frac{1}{1+x^2} \le \frac{1}{x}$ for $x \ge 1$	Right so far: $x \le 1 + x^2$ for every x.
"$\int_1^{\infty} \frac{dx}{x}$ diverges."	The habit that causes the mistake: knowing only that the bigger function's integral diverges.
So $\int_1^{\infty} \frac{dx}{1+x^2}$ diverges ✗	Being below something infinite says nothing: the smaller function's area could still be finite, and here it is, $\frac{\pi}{2} - \arctan 1 = \frac{\pi}{4} \approx 0.785$ (Problem 6). To prove divergence, show the integrand is above a divergent one; to prove convergence, below a convergent one, as $\frac{1}{1+x^2} \le \frac{1}{x^2}$ does.

5. Using the p > 1 rule near zero

The p-test is usually learned on $[1, \infty)$, and the threshold gets carried over unchanged.

$\int_0^1 \frac{dx}{x^2}$, which is improper at 0	Right so far: the setting of Problem 3.
"$p = 2 > 1$, so it converges, to $\frac{1}{p-1}$."	The habit that causes the mistake: the rule for $[1, \infty)$ used on $(0, 1]$.
$\int_0^1 \frac{dx}{x^2} = 1$ ✗	Near 0 the test is reversed: it converges only for $p < 1$. Here $\int_\varepsilon^1 \frac{dx}{x^2} = \frac{1}{\varepsilon} - 1$, which is 99 at $\varepsilon = 0.01$ and 999 at $\varepsilon = 0.001$: it diverges. A larger power decays faster at infinity but blows up faster at 0.

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