Implicit differentiation: differentiate both sides, then solve for dy/dx

Ten implicit derivatives: the slope and tangent line of a circle and an ellipse, a curve with an xy term, an exponential and a log, a trig relation, the folium of Descartes, the horizontal and vertical tangents of a tilted ellipse, a second derivative, the derivative of arcsin, and logarithmic differentiation of a product and quotient, with worked solutions and the mistakes of dropping dy/dx, skipping the product rule, differentiating a constant, losing a y' term and treating y as a constant.

BEFORE YOU START

Some curves are given by an equation in x and y that is hard or impossible to solve for y, such as $x^3 + y^3 = 6xy$. Implicit differentiation finds the slope anyway: treat y as a function of x, differentiate both sides of the equation with respect to x, and solve the result for y'. It is the same chain rule that drives related rates, and the idea behind differentiating through a quantity defined by an equation, such as the optimum of an inner problem, instead of by a formula. The work goes wrong in a few places: a y' dropped from the chain rule, a product rule skipped on xy, a constant side differentiated as if it were a variable, a y' term left on the wrong side, and y treated as a constant in a second derivative. These ten problems cover each of them.

- y is a function of x, and y' means $\frac{dy}{dx}$. Every term containing y is a composition, so its derivative carries a factor y': $\frac{d}{dx}\, y^2 = 2y\, y'$, $\frac{d}{dx}\, \sin y = \cos y\, y'$, $\frac{d}{dx}\, e^y = e^y\, y'$, $\frac{d}{dx}\, \ln y = \frac{y'}{y}$.
- A term with both variables needs the product rule as well: $\frac{d}{dx}(xy) = y + x\, y'$. A constant differentiates to 0.
- The method: (1) differentiate both sides of the equation with respect to x; (2) move every term containing y' to one side and everything else to the other; (3) factor out y' and divide. Substitute the point's coordinates at the end, or after step 1 if only a number is wanted.
- The slope usually depends on both x and y. The tangent line at (x_0, y_0) is $y - y_0 = m(x - x_0)$ with $m = y'$ there. A tangent is horizontal where $y' = 0$ and vertical where the denominator of y' is 0 and the numerator is not.
- Angles are in radians.

Builds on: Differentiation rules: chain, product, quotient

Problems

1. Find y' for the circle $x^2 + y^2 = 25$, and the tangent line at $(3, 4)$. difficulty ·

2. Find y' for the ellipse $x^2 + 4y^2 = 8$, and its value at $(2, 1)$ and at $(2, -1)$. difficulty ·

3. Find y' for $x^2 + 3xy + y^2 = 11$ at $(1, 2)$. difficulty ··

4. Find y' for $y\,e^x + \ln y = 1$ at the point where $x = 0$. difficulty ··

5. Find y' for $x \sin y + y = \pi$ at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$. difficulty ··

6. The folium of Descartes is $x^3 + y^3 = 6xy$. Find the slope and the tangent line at $(3, 3)$. difficulty ···

7. Find the points on $x^2 - xy + y^2 = 3$ where the tangent is horizontal, and where it is vertical. difficulty ···

8. For $x^2 + y^2 = 25$, find y'' in terms of y alone, and its value at $(3, 4)$. difficulty ···

9. Use $\sin y = x$ to find the derivative of $y = \arcsin x$, and its value at $x = \frac{1}{2}$. difficulty ··

10. Use logarithmic differentiation to find y' for $y = \dfrac{(x+1)^2(x+2)^3}{(x+3)^4}$, and its value at $x = 0$. difficulty ···

Answers

1. $y' = -\dfrac{x}{y}$; tangent $y = -\frac{3}{4}x + \frac{25}{4}$
2. $y' = -\dfrac{x}{4y}$; $-\frac{1}{2}$ at $(2,1)$ and $\frac{1}{2}$ at $(2,-1)$
3. $y' = -\dfrac{2+6}{3+4} = -\frac{8}{7}$
4. $y' = -\frac{1}{2}$
5. $y' = -\dfrac{1}{0+1} = -1$
6. Slope -1; tangent $y = -x + 6$
7. Horizontal at $(1,2)$ and $(-1,-2)$; vertical at $(2,1)$ and $(-2,-1)$
8. $y'' = -\dfrac{25}{y^3}$; at $(3,4)$, $-\frac{25}{64} \approx -0.391$
9. At $x = \frac{1}{2}$: $\dfrac{1}{\sqrt{3/4}} = \dfrac{2}{\sqrt{3}} \approx 1.155$
10. $y' = \frac{8}{81} \cdot \frac{13}{6} = \frac{52}{243} \approx 0.214$

Worked solutions

Problem 1

> Find y' for the circle $x^2 + y^2 = 25$, and the tangent line at $(3, 4)$.

$2x + 2y\,y' = 0$	Differentiate both sides; $\frac{d}{dx}\,y^2 = 2y\,y'$ by the chain rule, and 25 gives 0.
$y' = -\dfrac{x}{y}$	Move $2x$ across and divide by $2y$.
At $(3, 4)$: $y' = -\frac{3}{4}$	Substitute.
$y - 4 = -\frac{3}{4}(x - 3)$	Point-slope form.
$y' = -\dfrac{x}{y}$; **tangent** $y = -\frac{3}{4}x + \frac{25}{4}$	The tangent is perpendicular to the radius, whose slope is $\frac{4}{3}$.

Problem 2

> Find y' for the ellipse $x^2 + 4y^2 = 8$, and its value at $(2, 1)$ and at $(2, -1)$.

$2x + 8y\,y' = 0$	Differentiate both sides.
$y' = -\dfrac{x}{4y}$	Solve for y'.
At $(2, 1)$: $y' = -\frac{2}{4} = -\frac{1}{2}$. At $(2, -1)$: $y' = \frac{1}{2}$.	Same x, two points on the curve.
$y' = -\dfrac{x}{4y}$; $-\frac{1}{2}$ **at** $(2, 1)$ **and** $\frac{1}{2}$ **at** $(2, -1)$	The slope needs both coordinates, because the ellipse is not the graph of one function of x.

Problem 3

Find y' for $x^2 + 3xy + y^2 = 11$ at $(1, 2)$.

$1 + 6 + 4 = 11$	The point is on the curve.
$2x + (3y + 3x\,y') + 2y\,y' = 0$	The product rule on $3xy$ gives $3y + 3x\,y'$.
$y'(3x + 2y) = -(2x + 3y)$	Collect the y' terms on one side.
$y' = -\dfrac{2x + 3y}{3x + 2y}$	Divide.
$y' = -\dfrac{2 + 6}{3 + 4} = -\frac{8}{7}$	Substitute $x = 1$, $y = 2$.

Problem 4

Find y' for $y\,e^x + \ln y = 1$ at the point where $x = 0$.

At $x = 0$: $y + \ln y = 1$, so $y = 1$.	$1 + \ln 1 = 1$, and $y + \ln y$ is increasing, so 1 is the only solution.
$y'\,e^x + y\,e^x + \dfrac{y'}{y} = 0$	Product rule on $y\,e^x$; chain rule on $\ln y$.
$y'\left(e^x + \dfrac{1}{y}\right) = -y\,e^x$	Collect the y' terms.
$y'(1 + 1) = -1$	Substitute $x = 0$, $y = 1$.
$y' = -\frac{1}{2}$	Solving for y first is not possible in elementary functions, and it was never needed.

Problem 5

Find y' for $x \sin y + y = \pi$ at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$\frac{\pi}{2} \cdot 1 + \frac{\pi}{2} = \pi$	The point is on the curve.

$\sin y + x\cos y\, y' + y' = 0$	Product rule on $x \sin y$, with $\frac{d}{dx}\sin y = \cos y\, y'$; the constant π gives 0.
$y' = -\dfrac{\sin y}{x\cos y + 1}$	Collect and divide.
$y' = -\dfrac{1}{0+1} = -1$	$\sin\frac{\pi}{2} = 1$ and $\cos\frac{\pi}{2} = 0$.

Problem 6

> The folium of Descartes is $x^3 + y^3 = 6xy$. Find the slope and the tangent line at $(3, 3)$.

$27 + 27 = 54 = 6 \cdot 9$	The point is on the curve.
$3x^2 + 3y^2\, y' = 6y + 6x\, y'$	Chain rule on y^3; product rule on the right.
$y'(3y^2 - 6x) = 6y - 3x^2$	Both sides have a y' term: collect them on the left.
$y' = \dfrac{2y - x^2}{y^2 - 2x}$	Divide, and cancel the common factor 3.
$y' = \dfrac{6 - 9}{9 - 6} = -1$	Substitute.
Slope -1; **tangent** $y = -x + 6$	$y - 3 = -(x - 3)$. The curve is symmetric in x and y, and $(3, 3)$ is on the line of symmetry, so a slope of -1 is expected.

Problem 7

> Find the points on $x^2 - xy + y^2 = 3$ where the tangent is horizontal, and where it is vertical.

$2x - (y + x\, y') + 2y\, y' = 0$	Product rule on xy, with its minus sign.
$y' = \dfrac{y - 2x}{2y - x}$	$y'(2y - x) = y - 2x$.

Horizontal: $y = 2x$, so $x^2 - 2x^2 + 4x^2 = 3x^2 = 3$ and $x = \pm 1$.	Numerator 0; substitute $y = 2x$ into the curve.
Vertical: $x = 2y$, so $4y^2 - 2y^2 + y^2 = 3y^2 = 3$ and $y = \pm 1$.	Denominator 0; substitute $x = 2y$.
Horizontal at $(1, 2)$ **and** $(-1, -2)$; **vertical at** $(2, 1)$ **and** $(-2, -1)$	At each point the other part of the fraction is not 0: at $(1, 2)$ the denominator is 3, at $(2, 1)$ the numerator is -3.

Problem 8

> For $x^2 + y^2 = 25$, find y'' in terms of y alone, and its value at $(3, 4)$.

$y' = -\dfrac{x}{y}$	Problem 1.
$y'' = -\dfrac{1 \cdot y - x\,y'}{y^2}$	Quotient rule; y is still a function of x, so its derivative is y', not 0.
$y'' = -\dfrac{y + x^2/y}{y^2} = -\dfrac{y^2 + x^2}{y^3}$	Substitute $y' = -\frac{x}{y}$ and multiply top and bottom by y.
$y'' = -\dfrac{25}{y^3}$	Use the curve: $x^2 + y^2 = 25$.
$y'' = -\dfrac{25}{y^3}$; **at** $(3, 4)$, $-\frac{25}{64} \approx -0.391$	Negative on the upper half, where the circle is concave down.

Problem 9

> Use $\sin y = x$ to find the derivative of $y = \arcsin x$, and its value at $x = \frac{1}{2}$.

$\cos y \; y' = 1$	Differentiate $\sin y = x$.
$y' = \dfrac{1}{\cos y}$	Divide.

$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$	arcsin takes values in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, where $\cos y \geq 0$, so the positive root is the right one.
$\dfrac{d}{dx} \arcsin x = \dfrac{1}{\sqrt{1 - x^2}}$	For $-1 < x < 1$.
At $x = \frac{1}{2}$**:** $\dfrac{1}{\sqrt{3/4}} = \dfrac{2}{\sqrt{3}} \approx 1.155$	The same steps on $\tan y = x$ give $\frac{1}{1+x^2}$ for arctan.

Problem 10

> Use logarithmic differentiation to find y' for $y = \dfrac{(x+1)^2(x+2)^3}{(x+3)^4}$, and its value at $x = 0$.

$\ln y = 2\ln(x+1) + 3\ln(x+2) - 4\ln(x+3)$	Take logs (for $x > -1$, where every factor is positive): products become sums and powers become coefficients.
$\dfrac{y'}{y} = \dfrac{2}{x+1} + \dfrac{3}{x+2} - \dfrac{4}{x+3}$	Differentiate; the left side is a chain rule on $\ln y$.
$y' = y\left(\dfrac{2}{x+1} + \dfrac{3}{x+2} - \dfrac{4}{x+3}\right)$	Multiply by y, then write y out in full.
At $x = 0$: $y = \frac{1 \cdot 8}{81} = \frac{8}{81}$ and $\frac{y'}{y} = 2 + \frac{3}{2} - \frac{4}{3} = \frac{13}{6}$.	Substitute.
$y' = \frac{8}{81} \cdot \frac{13}{6} = \frac{52}{243} \approx 0.214$	The quotient rule gets the same answer through a much longer numerator.

Where this goes wrong

1. Dropping the y' on a y term

Differentiating y^2 gives $2y$ in every exercise where y is the variable.

$x^2 + 3xy + y^2 = 11$ at $(1, 2)$	Right so far: Problem 3.

"$\frac{d}{dx}\, y^2 = 2y$."	The habit that causes the mistake: differentiating y^2 as if y were the variable of differentiation.
$2x + 3y + 3x\,y' + 2y = 0$, so $y' = -\dfrac{2x + 5y}{3x} = -4$ ✗	y depends on x, so $\frac{d}{dx}\, y^2 = 2y\,y'$. With the factor, $y'(3x + 2y) = -(2x + 3y)$ and $y' = -\frac{8}{7}$. Every term with a y in it must produce a y'.

2. Skipping the product rule on xy

$3xy$ looks like a single term, so it gets a single derivative.

$x^2 + 3xy + y^2 = 11$ at $(1, 2)$	Right so far: Problem 3.
"$\frac{d}{dx}(3xy) = 3x\,y'$."	The habit that causes the mistake: treating $3x$ as a constant coefficient of y.
$2x + 3x\,y' + 2y\,y' = 0$, so $y' = -\dfrac{2x}{3x + 2y} = -\frac{2}{7}$ ✗	x changes too, so $3xy$ is a product: $\frac{d}{dx}(3xy) = 3y + 3x\,y'$. The missing $3y$ changes the numerator from $2x + 3y = 8$ to $2x = 2$, and the answer is $-\frac{8}{7}$, four times as steep.

3. Differentiating the constant side

The right side is π, and π looks like an angle, so it is treated like a variable.

$x \sin y + y = \pi$ at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$	Right so far: Problem 5.
"$\frac{d}{dx}\, \pi = 1$."	The habit that causes the mistake: differentiating every symbol that is not a digit.
$\sin y + x \cos y\, y' + y' = 1$, so $1 + y' = 1$ and $y' = 0$ ✗	π is a number, and the derivative of a number is 0. With 0 on the right, $1 + y' = 0$ and $y' = -1$. A horizontal tangent here would be a coincidence; check the answer against the curve instead.

4. Leaving a y' term on the other side

After differentiating, the left side has a y' and is solved for it, the way an explicit derivative would be.

$3x^2 + 3y^2\,y' = 6y + 6x\,y'$ at $(3, 3)$	Right so far: Problem 6, step 2.
"$y' = \dfrac{6y - 3x^2}{3y^2}$."	The habit that causes the mistake: isolating the y' on the left and treating the right side as known.
$y' = \dfrac{18 - 27}{27} = -\frac{1}{3}$ ✗	The right side still contains $6x\,y'$, so this is not a solution for y': y' appears on both sides. Collect the terms first, $y'(3y^2 - 6x) = 6y - 3x^2$, and the slope is $\frac{-9}{9} = -1$.

5. Treating y as a constant in the second derivative

In $-\frac{x}{y}$ the y sits in the denominator like a constant, so only the x gets differentiated.

$y' = -\dfrac{x}{y}$ on $x^2 + y^2 = 25$, at $(3, 4)$	Right so far: Problem 8, step 1.
"$y'' = -\dfrac{1}{y}$."	The habit that causes the mistake: forgetting that y is a function of x once the first derivative is done.
$y'' = -\frac{1}{4}$ at $(3, 4)$ ✗	The quotient rule needs the derivative of the denominator, which is $y' = -\frac{3}{4}$, not 0: $y'' = -\frac{y - x\,y'}{y^2} = -\frac{4 + 9/4}{16} = -\frac{25}{64}$. Then substitute y' and use the curve to simplify, as in Problem 8.

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