Gaussian elimination, inverses and determinants by hand

Ten problems on solving linear systems by row reduction: a 2×2 and a 3×3 system on the augmented matrix, a zero pivot that needs a row swap, a free variable and an inconsistent row, the 2×2 inverse formula, a 3×3 inverse by Gauss–Jordan on [A | I], a determinant by row reduction, the rules for det(cA), det(AB) and det(A^{-1}), an LU factorization with forward and back substitution, and a singular matrix whose null space is a redundant feature, with worked solutions and the mistakes of updating only the left side, dropping 1/det and forgetting the sign of a swap.

BEFORE YOU START

Almost every computation with matrices that is done by hand, and most of the ones a linear algebra library does for you, is Gaussian elimination underneath: solving $Ax = b$, inverting A, computing $\det A$, finding a rank or a null space. The method is three row operations used to clear the entries below each pivot, followed by back substitution. It goes wrong in a few predictable places: an operation applied to the coefficients but not to the right-hand side, a zero pivot divided by instead of swapped away, a row of zeros misread, the $\frac{1}{\det}$ dropped from the 2×2 inverse, and a row swap that silently flips the sign of a determinant. These ten problems cover each of them, and end with the singular matrix you meet in machine learning when one feature is the sum of two others.

- **The augmented matrix.** The system $Ax = b$ is written $[A \mid b]$: the coefficients of each equation in a row, with its right-hand side after the bar. A row operation changes the **whole** row, right-hand side included.
- **Row operations.** (1) Add a multiple of one row to another, written $R_2 \to R_2 - 3R_1$. (2) Swap two rows, $R_1 \leftrightarrow R_2$. (3) Multiply a row by a nonzero number. None of them changes the solutions of the system.
- **Pivots and echelon form.** Working left to right, the first nonzero entry of each row is its pivot; elimination uses each pivot to clear the entries below it. A zero where a pivot should be is fixed by swapping with a lower row. In echelon form each pivot is to the right of the one above, and any rows of zeros are at the bottom.
- **Reading the result.** A pivot in every column of A: one solution, found by back substitution from the last row up. A column without a pivot: its variable is free, and there are infinitely many solutions. A row $[0 \ \cdots \ 0 \mid c]$ with $c \neq 0$ says $0 = c$: no solution. A row of all zeros, $0 = 0$, says nothing.

- **Inverses.** A^{-1} satisfies $AA^{-1} = A^{-1}A = I$, and exists exactly when $\det A \neq 0$. For 2×2: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. In general, row reduce $[A \mid I]$ until the left half is I; the right half is then A^{-1}.
- **Determinants.** For 2×2, $ad - bc$. Adding a multiple of one row to another leaves $\det$ unchanged, a swap multiplies it by -1, and scaling a row by c multiplies it by c. The determinant of a triangular matrix is the product of its diagonal entries. For $n \times n$ matrices, $\det(AB) = \det A \, \det B$, $\det(A^{\top}) = \det A$ and $\det(cA) = c^n \det A$.
- **LU.** $A = LU$, with U the echelon form from elimination without swaps and L lower triangular with 1s on its diagonal and the multipliers ℓ_{ij} below it, where $R_i \to R_i - \ell_{ij} R_j$ was the operation used.

Problems

1. Solve $x + 2y = 5$, $3x + 4y = 6$ by row reducing the augmented matrix. difficulty ·

2. Solve the system $x + y + z = 6$, $2x + y - z = 1$, $x - 2y + 3z = 6$. difficulty ·

3. Solve $2y + z = 1$, $x + y + z = 4$, $2x + 3y + 2z = 7$. difficulty ··

4. Solve $x + 2y + z = 4$, $2x + 4y + 3z = 11$, $x + 2y + 2z = 7$. Then say what happens if the last right-hand side is 8 instead of 7. difficulty ··

5. Find the inverse of $A = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$ and use it to solve $Ax = (1, 2)^{\top}$. difficulty ·

6. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$ by row reducing $[A \mid I]$. difficulty ··

7. Compute $\det A$ for $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$ by row reduction, and check it by cofactor expansion. difficulty ··

8. A and B are 3×3 matrices with $\det A = 3$ and $\det B = -2$. Find $\det(2A)$, $\det(-A)$, $\det(A^{-1})$, $\det(A^{\top}A)$ and $\det(A^2 B)$. Can $\det(A+B)$ be found from this information? difficulty ··

9. Factor $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{bmatrix}$ as $A = LU$, use it to solve $Ax = (4, 10, 24)^{\top}$, and read off $\det A$. difficulty ···

10. A design matrix has rows $(1, 1, 2)$, $(1, 2, k)$ and $(1, 3, 4)$, one row per example and one column per feature. For which k is it singular? For that k, find its null space, solve $Ax = (2, 3, 4)^{\top}$, and show that $Ax = (1, 0, 0)^{\top}$ has no solution. difficulty ···

Answers

1. $x = -4$, $y = \frac{9}{2}$
2. $(x, y, z) = (1, 2, 3)$
3. $(x, y, z) = (2, -1, 3)$
4. $(x, y, z) = (1 - 2t,\ t,\ 3)$ for every t; with 8 the last row reads $0 = 1$ and there is no solution
5. $A^{-1} = \dfrac{1}{10}\begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$, $x = (-0.8, 0.6)^{\top}$
6. $A^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$
7. $\det A = -2$
8. 24, -3, $\frac{1}{3}$, 9, -18; $\det(A + B)$ cannot be found
9. $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, $x = (1, 1, 1)^{\top}$, $\det A = 2 \cdot 1 \cdot 2 = 4$
10. Singular for $k = 3$; null space $t(1, 1, -1)^{\top}$; $x = (t, t, 1 - t)^{\top}$ for $(2, 3, 4)^{\top}$; no solution for $(1, 0, 0)^{\top}$

Worked solutions

Problem 1

> Solve $x + 2y = 5$, $3x + 4y = 6$ by row reducing the augmented matrix.

$\left[\begin{array}{cc\|c} 1 & 2 & 5 \\ 3 & 4 & 6 \end{array}\right]$	Each equation is a row; the right-hand sides go after the bar.
$R_2 \to R_2 - 3R_1$: $\left[\begin{array}{cc\|c} 1 & 2 & 5 \\ 0 & -2 & -9 \end{array}\right]$ $\begin{bmatrix} \boxed{1} & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 5 \\ 0 & -2 & \boxed{-9} \end{bmatrix}$	The multiplier is $3 = \frac{3}{1}$, the entry to clear over the pivot. The right side changes too: $6 - 3 \cdot 5 = -9$.
$-2y = -9$, so $y = \frac{9}{2}$	The last row is one equation in one unknown.
$x + 2 \cdot \frac{9}{2} = 5$, so $x = -4$	Back substitution into the first row.
$x = -4$, $y = \frac{9}{2}$	Check in the second equation: $3(-4) + 4 \cdot \frac{9}{2} = -12 + 18 = 6$.

Problem 2

> Solve the system $x + y + z = 6$, $2x + y - z = 1$, $x - 2y + 3z = 6$.

$\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 6 \\ 2 & 1 & -1 & 1 \\ 1 & -2 & 3 & 6 \end{array}\right]$	The augmented matrix.
$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$: $\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 6 \\ 0 & -1 & -3 & -11 \\ 0 & -3 & 2 & 0 \end{array}\right]$	Clear the first column below the pivot 1.

$R_3 \to R_3 - 3R_2$: $\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 6 \\ 0 & -1 & -3 & -11 \\ 0 & 0 & 11 & 33 \end{array}\right]$	The second pivot is -1, and the multiplier is $\frac{-3}{-1} = 3$. Check the new row: $2 - 3(-3) = 11$ and $0 - 3(-11) = 33$.
$11z = 33$, so $z = 3$; $-y - 9 = -11$, so $y = 2$; $x + 2 + 3 = 6$, so $x = 1$	Back substitution from the bottom row up.
$(x, y, z) = (1, 2, 3)$	Check all three: $1 + 2 + 3 = 6$, $2 + 2 - 3 = 1$ and $1 - 4 + 9 = 6$.

Problem 3

> Solve $2y + z = 1$, $x + y + z = 4$, $2x + 3y + 2z = 7$.

$\left[\begin{array}{ccc\|c} 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 4 \\ 2 & 3 & 2 & 7 \end{array}\right]$	The first equation has no x, so the top-left entry is 0 and cannot be a pivot.
$R_1 \leftrightarrow R_2$: $\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 4 \\ 0 & 2 & 1 & 1 \\ 2 & 3 & 2 & 7 \end{array}\right]$	Swap in a row with a nonzero first entry. Swapping equations does not change the solutions.
$R_3 \to R_3 - 2R_1$: $\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 4 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{array}\right]$	Clear the first column.
$R_2 \leftrightarrow R_3$, then $R_3 \to R_3 - 2R_2$: $\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$	The swap is optional, but it makes the pivot 1 and avoids fractions; $R_3 \to R_3 - \frac{1}{2}R_2$ without it gives the same answer.
$(x, y, z) = (2, -1, 3)$	$z = 3$, $y = -1$, and $x = 4 - (-1) - 3 = 2$. Check: $2(-1) + 3 = 1$, $2 - 1 + 3 = 4$ and $4 - 3 + 6 = 7$.

Problem 4

Solve $x + 2y + z = 4$, $2x + 4y + 3z = 11$, $x + 2y + 2z = 7$. Then say what happens if the last right-hand side is 8 instead of 7.

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$: $\left[\begin{array}{ccc\|c} 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right]$	Clearing the first column also clears the second: the y coefficients are twice the x coefficients in every row.
$R_3 \to R_3 - R_2$: $\left[\begin{array}{ccc\|c} 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$	The last row says $0 = 0$, which is always true and adds no information.
The pivots are in the x and z columns, so y is free: $y = t$	No pivot in the y column means nothing pins y down.
$z = 3$ and $x = 4 - 2t - 3 = 1 - 2t$	Back substitution with $y = t$.
With 8: the same steps give the last row $[0\ 0\ 0 \mid 1]$	$R_3 - R_1$ now ends in $8 - 4 = 4$, and $R_3 - R_2$ in $4 - 3 = 1$.
$(x, y, z) = (1 - 2t,\ t,\ 3)$ **for every** t; **with** 8 **the last row reads** $0 = 1$ **and there is no solution**	Check $t = 1$: $(-1, 1, 3)$ gives $-1 + 2 + 3 = 4$, $-2 + 4 + 9 = 11$ and $-1 + 2 + 6 = 7$.

Problem 5

Find the inverse of $A = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$ and use it to solve $Ax = (1, 2)^{\top}$.

$\det A = 4 \cdot 6 - 7 \cdot 2 = 10$	Not zero, so A is invertible.
$A^{-1} = \dfrac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$	Swap the diagonal entries, negate the off-diagonal ones, divide by the determinant.

$AA^{-1} = \frac{1}{10}\begin{bmatrix} 24-14 & -28+28 \\ 12-12 & -14+24 \end{bmatrix} = I$	Check before using it.
$x = A^{-1}b = \frac{1}{10}\begin{bmatrix} 6-14 \\ -2+8 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -8 \\ 6 \end{bmatrix}$	Multiply both sides of $Ax = b$ on the left by A^{-1}.
$A^{-1} = \frac{1}{10}\begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$, $x = (-0.8, 0.6)^{\top}$	Check: $4(-0.8) + 7(0.6) = 1$ and $2(-0.8) + 6(0.6) = 2$.

Problem 6

Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$ by row reducing $[A \mid I]$.

$R_3 \to R_3 - 5R_1$: $\left[\begin{array}{ccc\|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{array}\right]$	Every operation acts on all six columns.
$R_3 \to R_3 + 4R_2$: third row $[0\ 0\ 1 \mid -5\ \ 4\ \ 1]$	$-15 + 16 = 1$, so the third pivot is already 1.
$R_2 \to R_2 - 4R_3$ and $R_1 \to R_1 - 3R_3$: rows $[0\ 1\ 0 \mid 20\ \ -15\ \ -4]$ and $[1\ 2\ 0 \mid 16\ \ -12\ \ -3]$	Now clear above the pivots, starting from the last column: $0 - 4(-5) = 20$, $1 - 16 = -15$, $1 + 15 = 16$.
$R_1 \to R_1 - 2R_2$: first row $[1\ 0\ 0 \mid -24\ \ 18\ \ 5]$	$16 - 40 = -24$, $-12 + 30 = 18$, $-3 + 8 = 5$.

$$\left[\begin{array}{ccc\|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc\|ccc} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

$[A \mid I]$ $\qquad$ $[I \mid A^{-1}]$

$A^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$	Check one entry of AA^{-1} per row: $(1,2,3)\cdot(-24,20,-5) = -24+40-15 = 1$, $(0,1,4)\cdot(18,-15,4) = 1$, $(5,6,0)\cdot(5,-4,1) = 1$. No row was scaled and no rows were swapped, so $\det A$ is the product of the pivots, 1, which is why the inverse has integer entries.

Problem 7

Compute $\det A$ for $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$ by row reduction, and check it by cofactor expansion.

$R_1 \leftrightarrow R_2$: $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 4 & -3 & 8 \end{bmatrix}$, and the determinant changes sign	The top-left entry is 0, so swap first. Keep a running factor of -1.
$R_3 \to R_3 - 4R_1$: third row $(0, -3, -4)$	Adding a multiple of a row does not change the determinant.
$R_3 \to R_3 + 3R_2$: third row $(0, 0, 2)$	$-4 + 6 = 2$; again no change.
$\det A = -1 \cdot (1 \cdot 1 \cdot 2) = -2$	The echelon form is triangular, so its determinant is the product of its diagonal; the one swap contributes the -1.

$$\underset{A}{\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} \boxed{1} & 0 & 3 \\ 0 & 1 & 2 \\ 4 & -3 & 8 \end{bmatrix} \xrightarrow{R_3 - 4R_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & \boxed{1} & 2 \\ 0 & -3 & -4 \end{bmatrix} \xrightarrow{R_3 + 3R_2} \begin{bmatrix} \boxed{1} & 0 & 3 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & \boxed{2} \end{bmatrix}$$

$\det A = -2$	Cofactor expansion along the first row: $0 - 1\,(1\cdot 8 - 3\cdot 4) + 2\,(1\cdot(-3) - 0\cdot 4) = 4 - 6 = -2$.

Problem 8

> A and B are 3×3 matrices with $\det A = 3$ and $\det B = -2$. Find $\det(2A)$, $\det(-A)$, $\det(A^{-1})$, $\det(A^{\top}A)$ and $\det(A^2 B)$. Can $\det(A+B)$ be found from this information?

$\det(2A) = 2^3 \det A = 24$	$2A$ doubles every one of the three rows, and each doubled row doubles the determinant.
$\det(-A) = (-1)^3 \det A = -3$	The same rule with $c = -1$; for an even size the sign would not change.
$\det(A^{-1}) = \dfrac{1}{\det A} = \dfrac{1}{3}$	$\det A \, \det(A^{-1}) = \det(AA^{-1}) = \det I = 1$.
$\det(A^{\top}A) = \det A^{\top} \det A = 9$ and $\det(A^2 B) = 3 \cdot 3 \cdot (-2) = -18$	The product rule, with $\det A^{\top} = \det A$.
24, -3, $\frac{1}{3}$, 9, -18; $\det(A+B)$ **cannot be found**	The determinant is not additive. With $A = \operatorname{diag}(3,1,1)$ and $B = \operatorname{diag}(-2,1,1)$, $\det(A+B) = 1 \cdot 2 \cdot 2 = 4$; with $B = \operatorname{diag}(1,-2,1)$ instead, $\det(A+B) = 4 \cdot (-1) \cdot 2 = -8$. Same determinants, different answers.

Problem 9

> Factor $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{bmatrix}$ as $A = LU$, use it to solve $Ax = (4, 10, 24)^{\top}$, and read off $\det A$.

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$: rows $(0,1,1)$ and $(0,3,5)$	Multipliers $\ell_{21} = 2$ and $\ell_{31} = 4$.
$R_3 \to R_3 - 3R_2$: row $(0,0,2)$, so $U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$	Multiplier $\ell_{32} = 3$.

$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ \boxed{2} & 1 & 0 \\ \boxed{4} & \boxed{3} & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{bmatrix}$ $L \qquad\qquad U \qquad\qquad A$	The multipliers go below the diagonal, in the positions they cleared, with their signs as used in $R_i \to R_i - \ell R_j$.
$Lc = b$: $c_1 = 4$, $c_2 = 10 - 2 \cdot 4 = 2$, $c_3 = 24 - 4 \cdot 4 - 3 \cdot 2 = 2$	Forward substitution, top row first. This repeats on b what elimination did to A.
$Ux = c$: $2x_3 = 2$, $x_2 + x_3 = 2$, $2x_1 + x_2 + x_3 = 4$	Back substitution: $x_3 = 1$, $x_2 = 1$, $x_1 = 1$.
$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, $x = (1, 1, 1)^\top$, $\det A = 2 \cdot 1 \cdot 2 = 4$	$\det A = \det L \det U$ and $\det L = 1$. Once L and U are known, each new right-hand side costs only the two triangular solves, which is why libraries factor once and reuse it.

Problem 10

A design matrix has rows $(1, 1, 2)$, $(1, 2, k)$ and $(1, 3, 4)$, one row per example and one column per feature. For which k is it singular? For that k, find its null space, solve $Ax = (2, 3, 4)^\top$, and show that $Ax = (1, 0, 0)^\top$ has no solution.

$\det A = 1\,(8 - 3k) - 1\,(4 - k) + 2\,(3 - 2) = 6 - 2k$	Cofactor expansion along the first row of $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & k \\ 1 & 3 & 4 \end{bmatrix}$.
$\det A = 0$ exactly when $k = 3$	Then the third column is $(2, 3, 4)^\top = (1, 1, 1)^\top + (1, 2, 3)^\top$: the third feature is the sum of the first two.

Row reduce with $k = 3$: $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_3 = t$, $x_2 = -t$, $x_1 = -t$; null space spanned by $(1, 1, -1)^\top$ $\underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}}_{A} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$R_2 - R_1$ and $R_3 - R_1$ give $(0, 1, 1)$ and $(0, 2, 2)$, and $R_3 - 2R_2$ gives zeros. Check: $A(1, 1, -1)^\top = (0, 0, 0)^\top$.
$(2, 3, 4)^\top$ is the third column, so $x = (0, 0, 1)^\top$ works, and every solution is $(0, 0, 1)^\top + t(1, 1, -1)^\top$	One solution plus anything in the null space.
For $(1, 0, 0)^\top$ the same operations end in the row $[0\ 0\ 0 \mid 1]$	The right side becomes $(1, -1, -1)$ after $R_2 - R_1$ and $R_3 - R_1$, and $R_3 - 2R_2$ turns its last entry into $-1 - 2(-1) = 1$. That row says $0 = 1$.
Singular for $k = 3$; null space $t(1, 1, -1)^\top$; $x = (t, t, 1 - t)^\top$ for $(2, 3, 4)^\top$; no solution for $(1, 0, 0)^\top$	In regression terms: weights w and $w + t(1, 1, -1)^\top$ give identical predictions, so $X^\top X$ is singular and the least-squares weights are not unique until a feature is dropped or a penalty such as ridge is added.

Where this goes wrong

1. Row operation applied to the coefficients only

The coefficients sit in a neat block, and the column after the bar is easy to leave behind.

$\left[\begin{array}{cc\|c} 1 & 2 & 5 \\ 3 & 4 & 6 \end{array}\right]$	Right so far: Problem 1.
"$R_2 \to R_2 - 3R_1$ turns $(3, 4)$ into $(0, -2)$."	The habit that causes the mistake: the operation is done on the coefficients and stops at the bar.

$\left[\begin{array}{cc\|c} 1 & 2 & 5 \\ 0 & -2 & 6 \end{array}\right]$, so $y = -3$ and $x = 11$ ✗	The right side must become $6 - 3 \cdot 5 = -9$. The check exposes it: $3 \cdot 11 + 4(-3) = 21$, not 6. The solution is $x = -4$, $y = \frac{9}{2}$.

2. Dropping the 1/det from the 2×2 inverse

The swap-and-negate pattern is the memorable part of the formula, and the scalar in front is not.

$A = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$, with $\begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$ from swapping and negating	Right so far: Problem 5.
"So $A^{-1} = \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$."	The step that causes the mistake: the division by $\det A = 10$ is skipped.
$x = \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} (1, 2)^\top = (-8, 6)^\top$ ✗	Then $Ax = (-32 + 42,\ -16 + 36)^\top = (10, 20)^\top$, ten times b: the matrix used is $10A^{-1}$, and A times it is $10I$, not I. Divide by $\det A$: $x = (-0.8, 0.6)^\top$.

3. A row of zeros read as no solution

A row with nothing left in it looks like a system that has broken down.

$\left[\begin{array}{ccc\|c} 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$	Right so far: Problem 4.
"The last row has no variables left."	True, and harmless: it says $0 = 0$.
"So the system has no solution." ✗	No solution needs a row $[0\ 0\ 0 \mid c]$ with $c \neq 0$. A row of zeros on both sides of the bar only says one equation was a combination of the others: here the third equation is the second minus the first. The y column has no pivot, so there are infinitely many solutions, $(1 - 2t, t, 3)$, such as $(1, 0, 3)$ and $(-1, 1, 3)$.

4. A row swap that keeps the determinant's sign

Row operations are thought of as harmless to the determinant, and for the most common one they are.

$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, swap $R_1 \leftrightarrow R_2$ to get a nonzero pivot	Right so far: Problem 7.
Eliminate to $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$	Also right: the two row additions do not change the determinant.
$\det A = 1 \cdot 1 \cdot 2 = 2$ ✗	A swap multiplies the determinant by -1, so $\det A = -2$, as cofactor expansion confirms. Two swaps would cancel; an odd number leaves a minus sign.

5. det(2A) taken as 2 det A

Scalars come out of most things linearly, so the 2 is pulled out once.

$\det A = 3$ for a 3×3 matrix A	Right so far: Problem 8.
"$2A$ is A times 2."	True; the question is what that does to the determinant.
$\det(2A) = 2 \cdot 3 = 6$ ✗	$2A$ multiplies each of the three rows by 2, and the determinant is linear in **each** row separately, so the factor comes out three times: $\det(2A) = 2^3 \cdot 3 = 24$. A quick test: $\det(2I) = 8$ for the 3×3 identity, not 2.

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