The fundamental theorem of calculus: derivatives of integrals and definite integrals

Ten problems on both parts of the fundamental theorem: an accumulation function checked against its closed form, a chain-rule upper limit, a variable lower limit, two moving limits and the maximum they give, critical points of an integral, definite integrals with an absolute value, the derivative of a Gaussian and an exponential CDF, and a limit of an integral by L'Hôpital's rule, with worked solutions and the mistakes of a missing chain-rule factor, a lost minus sign and confusing F with f.

BEFORE YOU START

The fundamental theorem is the link between the two halves of calculus: the derivative of an accumulated area is the height of the curve, and a definite integral is a difference of antiderivatives. In machine learning it is the reason a probability density is the derivative of its cumulative distribution function, so the density of any transformed variable comes from the chain rule on a CDF. The work goes wrong in a few places: a chain-rule factor left off a limit like x^3, the minus sign of a variable lower limit dropped, the value of the integrand at a constant lower limit subtracted, an absolute value integrated as if it were not there, and the accumulation function F confused with its integrand f. These ten problems cover each of them.

- **Part 1.** If f is continuous and $F(x) = \displaystyle\int_a^x f(t)\,dt$, then $F'(x) = f(x)$. The constant a does not appear in the answer.
- **Moving limits.** By the chain rule, $\dfrac{d}{dx}\displaystyle\int_{g(x)}^{h(x)} f(t)\,dt = f\big(h(x)\big)\,h'(x) - f\big(g(x)\big)\,g'(x)$. A constant limit contributes 0; a variable lower limit contributes a minus sign.
- **Part 2.** If f is continuous on $[a, b]$ and $F' = f$, then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$. Any antiderivative works, so no $+C$ is needed.
- $F(x) = \int_a^x f(t)\,dt$ is a function of x, not of t: t is a dummy variable. F is increasing where $f > 0$, so its critical points are the zeros of f.
- A **cumulative distribution function** is $F(x) = \displaystyle\int_{-\infty}^x p(t)\,dt$ for a density p, so $F' = p$. For the standard normal, $\Phi(z) = \displaystyle\int_{-\infty}^z \varphi(t)\,dt$ with $\varphi(t) = \dfrac{1}{\sqrt{2\pi}} e^{-t^2/2}$.
- $|x - c|$ is $c - x$ for $x < c$ and $x - c$ for $x > c$: split the integral at c.

Builds on: Differentiation rules: chain, product, quotient

Problems

1. Let $F(x) = \displaystyle\int_1^x (t^3 - 2t)\,dt$. Find $F'(x)$ by Part 1, then find $F(x)$ in closed form and check. difficulty ·

2. Find $\dfrac{d}{dx}\displaystyle\int_1^{x^3} \ln t\,dt$ for $x > 0$, and its value at $x = 2$. difficulty ··

3. Find $\dfrac{d}{dx}\displaystyle\int_x^5 \sqrt{1 + t^4}\,dt$, and its value at $x = 1$. difficulty ··

4. Let $G(x) = \displaystyle\int_x^{2x} e^{-t}\,dt$ for $x \geq 0$. Find $G'(x)$, and the x at which G is largest. difficulty ···

5. Let $F(x) = \displaystyle\int_0^x (t^2 - 4t + 3)\,dt$. Find the local maximum and minimum values of F. difficulty ··

6. Evaluate $\displaystyle\int_1^4 \left(2x + \frac{1}{\sqrt{x}}\right) dx$. difficulty ·

7. Evaluate $\displaystyle\int_0^3 |x - 1|\,dx$. difficulty ··

8. Φ is the standard normal CDF. Find $\dfrac{d}{dx}\Phi(2x - 1)$, and its value at $x = \frac{1}{2}$. difficulty ··

9. The waiting time T has density $p(t) = 2e^{-2t}$ for $t \geq 0$. Find its CDF $F(x) = P(T \leq x)$, check that $F' = p$, and find the median. difficulty ···

10. Find $\displaystyle\lim_{x \to 0} \frac{1}{x^3}\int_0^x \sin(t^2)\,dt$. difficulty ···

Answers

1. $F'(x) = x^3 - 2x$, and $F(x) = \frac{x^4}{4} - x^2 + \frac{3}{4}$ has that derivative
2. $9x^2 \ln x$; at $x = 2$ it is $36 \ln 2 \approx 24.95$
3. $-\sqrt{1 + x^4}$; at $x = 1$ it is $-\sqrt{2} \approx -1.414$
4. $G'(x) = 2e^{-2x} - e^{-x}$; G is largest at $x = \ln 2 \approx 0.693$, where $G = \frac{1}{4}$
5. Local maximum $\frac{4}{3}$ at $x = 1$; local minimum 0 at $x = 3$
6. 17
7. $\frac{5}{2}$
8. $2\varphi(2x - 1)$; at $x = \frac{1}{2}$ it is $\sqrt{2/\pi} \approx 0.798$
9. $F(x) = 1 - e^{-2x}$; median $\frac{\ln 2}{2} \approx 0.347$
10. $\frac{1}{3}$

Worked solutions

Problem 1

> Let $F(x) = \int_1^x (t^3 - 2t)\,dt$. Find $F'(x)$ by Part 1, then find $F(x)$ in closed form and check.

$F'(x) = x^3 - 2x$	Part 1: replace t with the upper limit.
$F(x) = \left[\frac{t^4}{4} - t^2\right]_1^x = \frac{x^4}{4} - x^2 - \left(\frac{1}{4} - 1\right)$	Part 2 with the antiderivative $\frac{t^4}{4} - t^2$.
$F(x) = \frac{x^4}{4} - x^2 + \frac{3}{4}$	The lower limit only adds the constant $\frac{3}{4}$.
$F'(x) = x^3 - 2x$, **and** $F(x) = \frac{x^4}{4} - x^2 + \frac{3}{4}$ **has that derivative**	The constant $\frac{3}{4}$ differentiates to 0, which is why the lower limit never appears in F'. Also $F(1) = 0$, as it must.

Problem 2

> Find $\dfrac{d}{dx}\displaystyle\int_1^{x^3} \ln t\,dt$ for $x > 0$, and its value at $x = 2$.

$\ln(x^3) \cdot \dfrac{d}{dx} x^3$	Put the upper limit into the integrand, then multiply by its derivative.
$= 3\ln x \cdot 3x^2$	$\ln(x^3) = 3\ln x$ and $(x^3)' = 3x^2$.
Check: $\displaystyle\int_1^{x^3} \ln t\,dt = \left[t\ln t - t\right]_1^{x^3} =$ $3x^3\ln x - x^3 + 1$, whose derivative is $9x^2\ln x + 3x^2 - 3x^2$	The long way agrees.
$9x^2\ln x$; **at** $x = 2$ **it is** $36\ln 2 \approx 24.95$	Without the chain-rule factor the answer would be $3\ln x$, off by a factor of $3x^2$.

Problem 3

> Find $\dfrac{d}{dx}\displaystyle\int_x^5 \sqrt{1+t^4}\,dt$, and its value at $x = 1$.

$\displaystyle\int_x^5 \sqrt{1+t^4}\,dt = -\int_5^x \sqrt{1+t^4}\,dt$	Swapping the limits changes the sign, and puts x in the upper limit.
$\dfrac{d}{dx}\Big(-\displaystyle\int_5^x \sqrt{1+t^4}\,dt\Big) = -\sqrt{1+x^4}$	Part 1.
$-\sqrt{1+x^4}$; **at** $x = 1$ **it is** $-\sqrt{2} \approx -1.414$	Moving the lower limit right removes area from a positive integrand, so the derivative must be negative.

Problem 4

> Let $G(x) = \displaystyle\int_x^{2x} e^{-t}\,dt$ for $x \geq 0$. Find $G'(x)$, and the x at which G is largest.

$G'(x) = e^{-2x}\cdot 2 - e^{-x}\cdot 1$	Upper limit $2x$ with derivative 2, minus the lower limit x with derivative 1.
$G'(x) = 0 \iff 2e^{-2x} = e^{-x} \iff e^{-x} = \frac{1}{2}$	Divide by $e^{-x} > 0$.
$x = \ln 2$	$G' > 0$ for $x < \ln 2$ and $G' < 0$ after, so this is the maximum.
$G(\ln 2) = e^{-\ln 2} - e^{-2\ln 2} = \frac{1}{2} - \frac{1}{4}$	$G(x) = e^{-x} - e^{-2x}$ by Part 2.
$G'(x) = 2e^{-2x} - e^{-x}$; G **is largest at** $x = \ln 2 \approx 0.693$, **where** $G = \frac{1}{4}$	$G(0) = 0$ and $G \to 0$ as $x \to \infty$: the interval $[x, 2x]$ starts empty and ends where e^{-t} is tiny.

Problem 5

> Let $F(x) = \displaystyle\int_0^x (t^2 - 4t + 3)\,dt$. Find the local maximum and minimum values of F.

$F'(x) = x^2 - 4x + 3 = (x-1)(x-3)$	Part 1. The critical points of F are the zeros of its integrand.

$F''(x) = 2x - 4$, so $F''(1) = -2$ and $F''(3) = 2$	Local maximum at 1, local minimum at 3.
$F(x) = \frac{x^3}{3} - 2x^2 + 3x$	Part 2; the lower limit 0 adds nothing.
$F(1) = \frac{1}{3} - 2 + 3 = \frac{4}{3}$ and $F(3) = 9 - 18 + 9 = 0$	The values of F, not of the integrand.
Local maximum $\frac{4}{3}$ at $x = 1$; local minimum 0 at $x = 3$	F climbs while the integrand is positive, on $(0, 1)$, and loses exactly that area again on $(1, 3)$, where the integrand is negative.

Problem 6

> Evaluate $\displaystyle\int_1^4 \left(2x + \frac{1}{\sqrt{x}}\right) dx$.

An antiderivative is $x^2 + 2\sqrt{x}$	$\int x^{-1/2}\,dx = \frac{x^{1/2}}{1/2} = 2\sqrt{x}$.
$\left(16 + 2 \cdot 2\right) - \left(1 + 2 \cdot 1\right) = 20 - 3$	$F(4) - F(1)$. The integrand is continuous on $[1, 4]$.
17	Term by term: $\int_1^4 2x\,dx = 15$ and $\int_1^4 x^{-1/2}\,dx = 2$.

Problem 7

> Evaluate $\displaystyle\int_0^3 |x - 1|\,dx$.

$\lvert x - 1\rvert = 1 - x$ on $[0, 1]$ and $x - 1$ on $[1, 3]$	Split where the inside changes sign.
$\displaystyle\int_0^1 (1 - x)\,dx = \left[x - \tfrac{x^2}{2}\right]_0^1 = \tfrac{1}{2}$	The first piece.

$\int_1^3 (x-1)\,dx = \left[\frac{x^2}{2} - x\right]_1^3 = \frac{3}{2} - \left(-\frac{1}{2}\right) = 2$	The second piece.
$\frac{5}{2}$	Two triangles: one with legs 1 and 1, one with legs 2 and 2.

Problem 8

Φ is the standard normal CDF. Find $\frac{d}{dx}\Phi(2x-1)$, and its value at $x = \frac{1}{2}$.

$\Phi(2x-1) = \int_{-\infty}^{2x-1} \varphi(t)\,dt$	A CDF is an integral with a variable upper limit.
$\frac{d}{dx}\Phi(2x-1) = \varphi(2x-1)\cdot 2$	Part 1 with the chain rule; the lower limit $-\infty$ does not move.
At $x = \frac{1}{2}$: $2\varphi(0) = \dfrac{2}{\sqrt{2\pi}}$	$\varphi(0) = \frac{1}{\sqrt{2\pi}}$.
$2\varphi(2x-1)$; **at** $x = \frac{1}{2}$ **it is** $\sqrt{2/\pi} \approx 0.798$	This is the density of $X = \frac{Z+1}{2}$: $P(X \le x) = \Phi(2x-1)$, and differentiating a CDF gives the density. It is the normal density with mean $\frac{1}{2}$ and standard deviation $\frac{1}{2}$.

Problem 9

The waiting time T has density $p(t) = 2e^{-2t}$ for $t \ge 0$. Find its CDF $F(x) = P(T \le x)$, check that $F' = p$, and find the median.

$F(x) = \int_0^x 2e^{-2t}\,dt = \Big[-e^{-2t}\Big]_0^x$	The density is 0 for $t < 0$, so the integral starts at 0.
$F(x) = 1 - e^{-2x}$ for $x \ge 0$	$-e^{-2x} - (-1)$. $F(0) = 0$ and $F \to 1$.
$F'(x) = 2e^{-2x} = p(x)$	Part 1, confirmed by differentiating the closed form.

$1 - e^{-2m} = \frac{1}{2} \iff e^{-2m} = \frac{1}{2} \iff m = \frac{\ln 2}{2}$	The median is where the CDF reaches $\frac{1}{2}$.
$F(x) = 1 - e^{-2x}$; **median** $\frac{\ln 2}{2} \approx 0.347$	The mean is $\frac{1}{2}$, larger than the median: the long right tail pulls it up.

Problem 10

> Find $\lim_{x \to 0} \frac{1}{x^3} \int_0^x \sin(t^2)\, dt$.

As $x \to 0$ both $\int_0^x \sin(t^2)\, dt$ and x^3 go to 0	The form is $\frac{0}{0}$, so L'Hôpital's rule applies.
$\lim_{x \to 0} \frac{\sin(x^2)}{3x^2}$	Differentiate the top by Part 1 and the bottom by the power rule.
$\frac{\sin(x^2)}{x^2} \to 1$	With $u = x^2 \to 0$, $\frac{\sin u}{u} \to 1$.
$\frac{1}{3}$	Near 0, $\sin(t^2) \approx t^2$, and $\int_0^x t^2\, dt = \frac{x^3}{3}$: the same answer without the rule.

Where this goes wrong

1. Leaving out the chain-rule factor

Part 1 is learned as "put the upper limit in", which is the whole rule only when the limit is x.

$\frac{d}{dx} \int_1^{x^3} \ln t\, dt$, $x > 0$	Right so far: Problem 2.
“Replace t with the upper limit.”	The habit that causes the mistake: the limit is x^3, not x, so it moves $3x^2$ times as fast.

$\ln(x^3) = 3\ln x$, so at $x = 2$ the derivative is $3\ln 2 \approx 2.079$ ✗	The edge of the area moves at the rate $(x^3)' = 3x^2$, and the derivative is $9x^2 \ln x$, which is $36\ln 2 \approx 24.95$ at $x = 2$, twelve times as much. Differentiating the closed form $3x^3 \ln x - x^3 + 1$ shows the factor.

2. Dropping the sign of a variable lower limit

In most problems x is the upper limit, so the minus sign is easy to forget.

$\dfrac{d}{dx}\displaystyle\int_x^5 \sqrt{1+t^4}\,dt$	Right so far: Problem 3.
"Put x into the integrand."	The habit that causes the mistake: Part 1 applied without first moving x to the upper limit.
The derivative is $\sqrt{1+x^4}$, so $\sqrt{2} \approx 1.414$ at $x = 1$ ✗	$\int_x^5 = -\int_5^x$, so the derivative is $-\sqrt{1+x^4}$, which is -1.414 at $x = 1$. As x increases, the interval $[x, 5]$ gets shorter and the integrand is positive, so the integral is decreasing.

3. Subtracting the integrand at the lower limit

Part 2 is "top minus bottom", and the pattern carries over to Part 1, where it does not belong.

$F(x) = \displaystyle\int_1^x (t^3 - 2t)\,dt$	Right so far: Problem 1.
"The derivative is f at the top minus f at the bottom."	The habit that causes the mistake: Part 2's subtraction copied into Part 1.
$F'(x) = (x^3 - 2x) - (1 - 2) = x^3 - 2x + 1$ ✗	The lower limit is a constant, and it contributes a constant, $\frac{3}{4}$, to F, which differentiates to 0. $F'(x) = x^3 - 2x$; at $x = 2$ that is 4, not 5. Only a lower limit that moves with x contributes to the derivative.

4. Integrating an absolute value without splitting

The bars look like decoration once the integral is set up.

$\int_0^3 \lvert x-1\rvert \, dx$	Right so far: Problem 7.
"For most of the interval $x - 1 \geq 0$, so use $x - 1$."	The habit that causes the mistake: one formula used on both sides of $x = 1$.
$\int_0^3 (x-1)\, dx = \frac{9}{2} - 3 = \frac{3}{2}$ ✗	On $[0, 1]$, $x - 1$ is negative, so the triangle there, of area $\frac{1}{2}$, is subtracted instead of added. Split at $x = 1$: $\frac{1}{2} + 2 = \frac{5}{2}$. The integral of a function that is never negative is a total area, and here $\frac{3}{2} = 2 - \frac{1}{2}$ is a net one.

5. Reporting the integrand's value instead of F's

F and its integrand share the same critical points, which makes it easy to evaluate the wrong one.

$F(x) = \int_0^x (t^2 - 4t + 3)\, dt$, local maximum at $x = 1$	Right so far: Problem 5, step 2.
"Plug $x = 1$ into the function."	The habit that causes the mistake: the function on the page is the integrand.
The local maximum value is $1 - 4 + 3 = 0$ ✗	0 is the integrand at $x = 1$: it is $F'(1)$, which is 0 at every critical point. The value of F is the area accumulated up to 1, $F(1) = \frac{1}{3} - 2 + 3 = \frac{4}{3}$. The local minimum value is $F(3) = 0$, which happens to be 0 for a different reason.

More at eulerpad.app. This is one of 82 free practice sets with worked solutions at eulerpad.app/practice, alongside guides to the topics behind them.

EulerPad, the app, reads your handwritten math on iPad, iPhone and Android and checks it line by line. It's free to start.

This set is also Chapter 12 of *Calculus I & II*, a 250-problem workbook with worked solutions: eulerpad.app/workbooks

eulerpad.app/practice/fundamental-theorem-of-calculus