Fourier series: coefficients, even and odd extensions, convergence and Parseval

Ten problems on Fourier series: the orthogonality integrals of sines and cosines on [−π, π], the square wave with its Gibbs overshoot, the sawtooth f(x) = x and its value at the jump, the cosine series of |x| and the sum of 1/(2k − 1)², the series of x² and the Basel sum π²/6, a function of period 4 with the coefficients over L, the half-range sine series of x(π − x) and a sum of alternating cubes, Parseval's identity for the sum of 1/n⁴, the complex form of e^x and the alternating sum of 1/(1 + n²), and the periodic response of y″ + 4y to a square wave, solved term by term, with worked solutions and the mistakes of writing a0 instead of a0/2, dividing by π when the period is 2L, taking the series to equal f at a jump, treating cos nπ as 0 and leaving the a0 term out of Parseval's identity.

BEFORE YOU START

A Fourier series writes a periodic function as a sum of sines and cosines whose frequencies are whole multiples of one base frequency. Where a Taylor series matches a function at one point, a Fourier series matches it over a whole period, and it works even for functions with corners and jumps, such as a square wave, that have no Taylor series at all. The coefficients come from one fact: on a full period, the sines and cosines are orthogonal, so integrating the function against $\cos nx$ or $\sin nx$ picks out exactly one coefficient, the way a dot product with a unit vector picks out one coordinate. Symmetry halves the work, since an even function has only cosines and an odd one only sines. At a jump the series lands on the average of the two sides, and Parseval's identity turns the energy of the function into a sum of squared coefficients, which is how series such as $\sum 1/n^2 = \pi^2/6$ get evaluated. The same expansion, in complex form, is the discrete Fourier transform behind signal processing and spectral methods. These ten problems compute coefficients, use symmetry and general periods, decide what the series converges to and solve a forced oscillator one frequency at a time.

- **Period** 2π. $f(x) \sim \frac{a_0}{2} + \sum_{n\geq 1}(a_n \cos nx + b_n \sin nx)$ with $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$. The constant term is $\frac{a_0}{2}$, the average of f over a period.
- **Orthogonality.** On $[-\pi, \pi]$, $\int \sin mx \sin nx\,dx$ and $\int \cos mx \cos nx\,dx$ are 0 for $m \neq n$ and π for $m = n \geq 1$, and $\int \sin mx \cos nx\,dx = 0$ always.
- **Period** $2L$. Replace nx by $\frac{n\pi x}{L}$ and $\frac{1}{\pi}\int_{-\pi}^{\pi}$ by $\frac{1}{L}\int_{-L}^{L}$.
- **Symmetry.** If f is even, every $b_n = 0$ and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$. If f is odd, every $a_n = 0$ and $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$. A function given only on $[0, \pi]$ can be extended either way: the odd extension gives the half-range sine series, the even one the half-range cosine series.
- **Values at multiples of** π. $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$.

- **Convergence (Dirichlet).** For a piecewise smooth f, the series converges to $f(x)$ where f is continuous and to the average $\frac{1}{2}\big(f(x^-)+f(x^+)\big)$ at a jump, including the jump where the periodic extension wraps around. Near a jump the partial sums overshoot by about 9% of the jump (the Gibbs phenomenon), however many terms are kept.
- **Parseval.** $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \frac{a_0^2}{2} + \sum_{n\ge 1}(a_n^2+b_n^2)$.
- **Complex form.** $f(x)\sim\sum_{n=-\infty}^{\infty} c_n e^{inx}$ with $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$. For real f, $c_n = \frac{1}{2}(a_n - ib_n)$ and $c_{-n}=\overline{c_n}$.

Builds on: Integration: substitution and parts, Trig identities, Series convergence tests: geometric, divergence, integral, comparison and alternating

Problems

1. Compute $\int_{-\pi}^{\pi}\sin 3x\sin 5x\,dx$, $\int_{-\pi}^{\pi}\sin^2 4x\,dx$ and $\int_{-\pi}^{\pi}\cos 2x\sin 7x\,dx$, and explain why each comes out as it does. difficulty ·

2. Find the Fourier series of the square wave of period 2π, difficulty ·

$$f(x)=\begin{cases}-1 & -\pi<x<0\\ 1 & 0<x<\pi,\end{cases}$$

and write out its first three nonzero terms.

3. Find the Fourier series of $f(x)=x$ on $-\pi<x<\pi$, extended with period 2π. What does the series converge to at $x=\pi/2$ and at $x=\pi$? difficulty ··

4. Find the Fourier series of $f(x)=|x|$ on $[-\pi,\pi]$, and use it at $x=0$ to find $\sum_{k\ge 1}\frac{1}{(2k-1)^2}$. difficulty ··

5. Find the Fourier series of $f(x)=x^2$ on $[-\pi,\pi]$. Evaluate it at $x=\pi$ to find $\sum_{n\ge 1}\frac{1}{n^2}$, and at $x=0$ to find $\sum_{n\ge 1}\frac{(-1)^{n+1}}{n^2}$. difficulty ··

6. Let f have period 4, with $f(x)=0$ for $-2<x<0$ and $f(x)=x$ for $0<x<2$. Find its Fourier series, and the value the series converges to at $x=2$. difficulty ···

7. Find the half-range sine series of $f(x) = x(\pi - x)$ on $[0, \pi]$, and evaluate it at $x = \pi/2$ to find $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$. difficulty ··

8. Apply Parseval's identity to the series of x^2 from Problem 5 to find $\sum_{n \geq 1} \frac{1}{n^4}$. difficulty ···

9. Find the complex Fourier coefficients c_n of $f(x) = e^x$ on $-\pi < x < \pi$, extended with period 2π. Use the series at $x = 0$ to find $\sum_{n \geq 1} \frac{(-1)^n}{1+n^2}$. difficulty ···

10. Find the periodic solution of $y'' + 4y = f(x)$, where f is the square wave of Problem 2, as a Fourier series, and give its first three nonzero terms. difficulty ···

Answers

1. 0, π and 0
2. $f(x) \sim \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$
3. $x \sim 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right)$; it converges to $\frac{\pi}{2}$ at $x = \frac{\pi}{2}$ and to 0 at $x = \pi$
4. $|x| \sim \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right)$, and $\sum \frac{1}{(2k-1)^2} = \frac{\pi^2}{8} \approx 1.2337$
5. $\sum \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.6449$ and $\sum \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} \approx 0.8225$
6. At $x = 2$ the series converges to 1
7. $x(\pi - x) \sim \frac{8}{\pi}\left(\sin x + \frac{1}{27}\sin 3x + \frac{1}{125}\sin 5x + \cdots\right)$, and the sum is $\frac{\pi^3}{32} \approx 0.9689$
8. $\sum \frac{1}{n^4} = \frac{\pi^4}{90} \approx 1.0823$
9. $c_n = \frac{(-1)^n \sinh \pi}{\pi(1 - in)}$, and $\sum \frac{(-1)^n}{1+n^2} = \frac{1}{2}\left(\frac{\pi}{\sinh \pi} - 1\right) \approx -0.3640$
10. $y = \frac{4}{3\pi}\sin x - \frac{4}{15\pi}\sin 3x - \frac{4}{105\pi}\sin 5x - \cdots$

Worked solutions

Problem 1

> Compute $\int_{-\pi}^{\pi} \sin 3x \sin 5x \, dx$, $\int_{-\pi}^{\pi} \sin^2 4x \, dx$ and $\int_{-\pi}^{\pi} \cos 2x \sin 7x \, dx$, and explain why each comes out as it does.

$\sin 3x \sin 5x = \frac{1}{2}(\cos 2x - \cos 8x)$	The product-to-sum identity $\sin A \sin B = \frac{1}{2}\big(\cos(A-B) - \cos(A+B)\big)$.
$\int_{-\pi}^{\pi} \cos kx \, dx = 0$ for $k \neq 0$	Over a whole number of periods a cosine has as much area above the axis as below.
$\sin^2 4x = \frac{1}{2}(1 - \cos 8x)$	The same identity with $A = B$: the $\cos 0 = 1$ term survives.
$\int_{-\pi}^{\pi} \frac{1}{2} \, dx = \pi$	Only the constant contributes.
0, π and 0	The last integrand $\cos 2x \sin 7x$ is odd, so its integral over $[-\pi, \pi]$ is 0 without any work. These three facts are why integrating against $\sin nx$ picks out b_n alone.

Problem 2

> Find the Fourier series of the square wave of period 2π,
>
> $$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi, \end{cases}$$
>
> and write out its first three nonzero terms.

$a_n = 0$ for every n	f is odd, so $f(x)\cos nx$ is odd and integrates to 0; that includes a_0.
$b_n = \frac{2}{\pi}\int_0^{\pi} \sin nx \, dx$	For an odd function, twice the integral over $[0, \pi]$.

$= \frac{2}{n\pi}\big(1 - \cos n\pi\big) = \frac{2}{n\pi}\big(1 - (-1)^n\big)$	$\big[-\frac{\cos nx}{n}\big]_0^{\pi}$.
$b_n = \frac{4}{n\pi}$ for odd n, $b_n = 0$ for even n	$1 - (-1)^n$ is 2 or 0.
$f(x) \sim \frac{4}{\pi}\big(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\big)$	The coefficients shrink only like $1/n$, the mark of a jump. Near each jump the partial sums peak at about 1.179 instead of 1: an overshoot of about 0.179, or 9% of the jump of 2, which moves closer to the jump but does not shrink as terms are added.

Problem 3

> Find the Fourier series of $f(x) = x$ on $-\pi < x < \pi$, extended with period 2π. What does the series converge to at $x = \pi/2$ and at $x = \pi$?

$a_n = 0$	x is odd.
$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx\, dx$	Twice the half-period integral.
$\int_0^{\pi} x \sin nx\, dx = \Big[-\frac{x\cos nx}{n}\Big]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\, dx$	Parts with $u = x$, $dv = \sin nx\, dx$. The last integral is 0.
$= -\frac{\pi(-1)^n}{n}$, so $b_n = \frac{2(-1)^{n+1}}{n}$	$\cos n\pi = (-1)^n$.
$x \sim 2\big(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\big)$; **it converges to $\frac{\pi}{2}$ at $x = \frac{\pi}{2}$ and to 0 at $x = \pi$**	At π the periodic extension jumps from π down to $-\pi$, and the series takes the average, 0: every term has $\sin n\pi = 0$ (Mistake 3). At $\pi/2$ the series reads $2(1 - \frac{1}{3} + \frac{1}{5} - \cdots) = \frac{\pi}{2}$, which is Leibniz's $1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}$.

Problem 4

> Find the Fourier series of $f(x) = |x|$ on $[-\pi, \pi]$, and use it at $x = 0$ to find $\sum_{k \geq 1} \frac{1}{(2k-1)^2}$.

$b_n = 0$ and $a_0 = \frac{2}{\pi}\int_0^\pi x\,dx = \pi$	$\lvert x\rvert$ is even. The constant term is $\frac{a_0}{2} = \frac{\pi}{2}$, its average.
$a_n = \frac{2}{\pi}\int_0^\pi x\cos nx\,dx = \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^\pi$	Integration by parts.
$a_n = \frac{2\left((-1)^n - 1\right)}{\pi n^2}$	The sin term vanishes at both ends; $\cos n\pi = (-1)^n$ and $\cos 0 = 1$.
$a_n = -\frac{4}{\pi n^2}$ for odd n, 0 for even n	Mistaking $\cos n\pi$ for 0 gives $-\frac{2}{\pi n^2}$ for every n (Mistake 4).
$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k \geq 1} \frac{1}{(2k-1)^2}$	At $x = 0$ the function is continuous, so the series equals $\lvert 0\rvert = 0$.
$\lvert x\rvert \sim \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right)$, **and** $\sum \frac{1}{(2k-1)^2} = \frac{\pi^2}{8} \approx 1.2337$	The coefficients fall like $1/n^2$, faster than the square wave's: $\lvert x\rvert$ has corners but no jumps.

Problem 5

> Find the Fourier series of $f(x) = x^2$ on $[-\pi, \pi]$. Evaluate it at $x = \pi$ to find $\sum_{n \geq 1} \frac{1}{n^2}$, and at $x = 0$ to find $\sum_{n \geq 1} \frac{(-1)^{n+1}}{n^2}$.

$a_0 = \frac{2}{\pi}\int_0^\pi x^2\,dx = \frac{2\pi^2}{3}$	Even function, so $b_n = 0$. The constant term is $\frac{a_0}{2} = \frac{\pi^2}{3}$.
$a_n = \frac{2}{\pi}\int_0^\pi x^2\cos nx\,dx = \frac{4(-1)^n}{n^2}$	Integrate by parts twice; only the term $\frac{2x\cos nx}{n^2}$ survives at the ends.
$x^2 \sim \frac{\pi^2}{3} + 4\sum_{n \geq 1} \frac{(-1)^n}{n^2}\cos nx$	Assemble the series.
At $x = \pi$: $\pi^2 = \frac{\pi^2}{3} + 4\sum \frac{1}{n^2}$	$(-1)^n \cos n\pi = 1$. The periodic extension of x^2 is continuous at π, so the series equals π^2 there.

At $x = 0$: $0 = \frac{\pi^2}{3} - 4\sum \frac{(-1)^{n+1}}{n^2}$	$(-1)^n \cos 0 = (-1)^n$.
$\sum \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.6449$ **and** $\sum \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} \approx 0.8225$	The Basel sum. Writing a_0 instead of $\frac{a_0}{2}$ as the constant term breaks both evaluations (Mistake 1).

Problem 6

Let f have period 4, with $f(x) = 0$ for $-2 < x < 0$ and $f(x) = x$ for $0 < x < 2$. Find its Fourier series, and the value the series converges to at $x = 2$.

$L = 2$, so the terms are $\cos \frac{n\pi x}{2}$ and $\sin \frac{n\pi x}{2}$	The period is $2L = 4$.
$a_0 = \frac{1}{2}\int_0^2 x\,dx = 1$	$\frac{1}{L}\int_{-L}^{L}$, and $f = 0$ on the left half. The constant term $\frac{a_0}{2} = \frac{1}{2}$ is the average of f over a period.
$a_n = \frac{1}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx = \frac{2\left((-1)^n - 1\right)}{n^2\pi^2}$	Parts with $k = \frac{n\pi}{2}$: $\left[\frac{x\sin kx}{k} + \frac{\cos kx}{k^2}\right]_0^2 = \frac{(-1)^n - 1}{k^2}$.
$b_n = \frac{1}{2}\int_0^2 x\sin\frac{n\pi x}{2}\,dx = \frac{2(-1)^{n+1}}{n\pi}$	$\frac{1}{2}\left[-\frac{x\cos kx}{k}\right]_0^2 = -\frac{(-1)^n}{k}$; the $\sin kx$ term vanishes.
$f \sim \frac{1}{2} - \frac{4}{\pi^2}\sum_{n\text{ odd}} \frac{1}{n^2}\cos\frac{n\pi x}{2} + \frac{2}{\pi}\sum_{n\geq 1}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{2}$	The cosine terms are the even part of f, the sine terms the odd part.
At $x = 2$ the series converges to 1	f jumps from 2 down to 0 there, and $\frac{1}{2}(2 + 0) = 1$. Check: $\cos n\pi = -1$ for odd n gives $\frac{1}{2} + \frac{4}{\pi^2}\cdot\frac{\pi^2}{8} = 1$, using Problem 4. Using $\frac{1}{\pi}$ in place of $\frac{1}{L}$ gives the wrong average (Mistake 2).

Problem 7

Find the half-range sine series of $f(x) = x(\pi - x)$ on $[0, \pi]$, and evaluate it at $x = \pi/2$ to find $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$.

Extend f to $[-\pi, 0]$ as an odd function	Then the series has sines only, and on $[0, \pi]$ it represents f itself.

$b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx\,dx$	The odd-function formula.
$b_n = \frac{4\left(1-(-1)^n\right)}{\pi n^3}$	Parts three times, or split as $\pi x - x^2$ and reuse the integrals of Problems 3 and 5.
$b_n = \frac{8}{\pi n^3}$ for odd n, 0 for even n	The odd extension is smooth enough that its coefficients fall like $1/n^3$.
$\frac{\pi^2}{4} = \frac{8}{\pi}\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right)$	At $x = \frac{\pi}{2}$, $f = \frac{\pi^2}{4}$, and $\sin\frac{n\pi}{2}$ is $1, -1, 1, \ldots$ for $n = 1, 3, 5, \ldots$
$x(\pi - x) \sim \frac{8}{\pi}\left(\sin x + \frac{1}{27}\sin 3x + \frac{1}{125}\sin 5x + \cdots\right)$, **and the sum is** $\frac{\pi^3}{32} \approx 0.9689$	The first term alone already has nearly the right height: its peak $\frac{8}{\pi} \approx 2.546$ is close to the true peak $\frac{\pi^2}{4} \approx 2.467$.

Problem 8

> Apply Parseval's identity to the series of x^2 from Problem 5 to find $\sum_{n\ge1}\frac{1}{n^4}$.

$\frac{1}{\pi}\int_{-\pi}^{\pi} x^4\,dx = \frac{2\pi^4}{5}$	The left side of Parseval: $\frac{1}{\pi}\cdot\frac{2\pi^5}{5}$.
$\frac{a_0^2}{2} = \frac{1}{2}\left(\frac{2\pi^2}{3}\right)^2 = \frac{2\pi^4}{9}$	The constant term's share, with $a_0 = \frac{2\pi^2}{3}$.
$\sum a_n^2 = \sum \frac{16}{n^4}$	$a_n = \frac{4(-1)^n}{n^2}$ and $b_n = 0$.
$16\sum\frac{1}{n^4} = \frac{2\pi^4}{5} - \frac{2\pi^4}{9} = \frac{8\pi^4}{45}$	Solve for the sum.
$\sum\frac{1}{n^4} = \frac{\pi^4}{90} \approx 1.0823$	Leaving out the $\frac{a_0^2}{2}$ term gives $\frac{\pi^4}{40} \approx 2.435$ (Mistake 5).

Problem 9

> Find the complex Fourier coefficients c_n of $f(x) = e^x$ on $-\pi < x < \pi$, extended with period 2π. Use the series at $x = 0$ to find $\sum_{n\ge1}\frac{(-1)^n}{1+n^2}$.

$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)x}\,dx$	Combine e^x and e^{-inx}.

$= \frac{e^{(1-in)\pi} - e^{-(1-in)\pi}}{2\pi(1-in)}$	The exponential rule with a complex constant.
$e^{\mp in\pi} = (-1)^n$	So the numerator is $(-1)^n(e^{\pi} - e^{-\pi}) = 2(-1)^n \sinh \pi$.
$c_n = \frac{(-1)^n \sinh \pi}{\pi(1-in)}$	For every integer n, including $n = 0$, where $c_0 = \frac{\sinh \pi}{\pi}$ is the average of e^x.
$c_n + c_{-n} = \frac{(-1)^n \sinh \pi}{\pi} \cdot \frac{2}{1+n^2}$	At $x = 0$ every $e^{inx} = 1$; pair n with $-n$, since $\frac{1}{1-in} + \frac{1}{1+in} = \frac{2}{1+n^2}$.
$1 = \frac{\sinh \pi}{\pi}\left(1 + 2\sum_{n \geq 1} \frac{(-1)^n}{1+n^2}\right)$	e^x is continuous at 0, so the series equals $e^0 = 1$.
$c_n = \frac{(-1)^n \sinh \pi}{\pi(1-in)}$, **and** $\sum \frac{(-1)^n}{1+n^2} = \frac{1}{2}\left(\frac{\pi}{\sinh \pi} - 1\right) \approx -0.3640$	At $x = \pi$ the series converges instead to $\cosh \pi$, the average of $e^{-\pi}$ and e^{π}.

Problem 10

> Find the periodic solution of $y'' + 4y = f(x)$, where f is the square wave of Problem 2, as a Fourier series, and give its first three nonzero terms.

Try $y = \sum_{n \text{ odd}} B_n \sin nx$	The forcing has only odd sine terms, and the equation maps $\sin nx$ to a multiple of itself.
$(\sin nx)'' + 4 \sin nx = (4 - n^2) \sin nx$	Differentiate twice.
$(4 - n^2)B_n = \frac{4}{n\pi}$ for odd n	Match the coefficients of $\sin nx$ on both sides.
$B_n = \frac{4}{n\pi(4-n^2)}$	No odd n makes $4 - n^2 = 0$: the forcing has no $\sin 2x$ term, so there is no resonance with the natural frequency 2.
$B_1 = \frac{4}{3\pi} \approx 0.4244$, $B_3 = -\frac{4}{15\pi} \approx -0.0849$, $B_5 = -\frac{4}{105\pi} \approx -0.0121$	$4 - n^2$ is 3, -5, -21.

$y = \frac{4}{3\pi}\sin x - \frac{4}{15\pi}\sin 3x - \frac{4}{105\pi}\sin 5x - \cdots$	The B_n fall like $1/n^3$, so the response is far smoother than the square wave that drives it. Terms below the natural frequency stay in phase with the forcing; those above it come out with the opposite sign.

Where this goes wrong

1. a0 written as the constant term instead of a0/2

The constant term is taken straight from the coefficient formula.

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2\pi^2}{3}$	Right so far: Problem 5.
"$x^2 \sim \frac{2\pi^2}{3} + 4\sum \frac{(-1)^n}{n^2}\cos nx$."	The formula for a_n with $n = 0$ has a $\frac{1}{\pi}$ where the average needs $\frac{1}{2\pi}$, which is why the series starts with $\frac{a_0}{2}$.
At $x = 0$: $0 = \frac{2\pi^2}{3} - 4 \cdot \frac{\pi^2}{12}$ ✗	The right side is $\frac{\pi^2}{3}$, not 0: the series is shifted up by $\frac{\pi^2}{3}$ everywhere. With the correct constant $\frac{\pi^2}{3}$ it gives 0, and at $x = \pi$ the shifted series gives $\frac{4\pi^2}{3}$ instead of π^2.

2. Dividing by π when the period is 2L

The coefficient formulas for period 2π are used on a function of period 4.

$f = x$ on $(0, 2)$, 0 on $(-2, 0)$, period 4	Right so far: Problem 6.
"$a_0 = \frac{1}{\pi}\int_0^2 x\,dx = \frac{2}{\pi}$."	For period $2L$ the factor is $\frac{1}{L} = \frac{1}{2}$, and the arguments are $\frac{n\pi x}{L}$: the π belongs inside the sines and cosines, not in front.
The constant term is $\frac{a_0}{2} = \frac{1}{\pi} \approx 0.318$ ✗	The average of f over a period is $\frac{1}{4}\int_0^2 x\,dx = \frac{1}{2}$, so the right a_0 is 1. Even with the right arguments $\frac{n\pi x}{2}$, every other coefficient is off by the same factor $\frac{2}{\pi}$.

3. The series taken to equal f at a jump

The series of x on $(-\pi, \pi)$ is evaluated at $x = \pi$ by plugging π into x.

$x \sim 2\sum \frac{(-1)^{n+1}}{n} \sin nx$ on $(-\pi, \pi)$	Right so far: Problem 3.
"At $x = \pi$ the function is π."	The series represents the periodic extension, which jumps from π to $-\pi$ at $x = \pi$.
The series converges to π at $x = \pi$ ✗	Every term is $\sin n\pi = 0$, so the series is 0 there: the average $\frac{1}{2}(\pi + (-\pi))$, as Dirichlet's theorem says. It equals x only strictly inside $(-\pi, \pi)$.

4. cos nπ taken as 0

In the coefficients of $|x|$, the endpoint terms are evaluated with $\cos n\pi$ confused with $\sin n\pi$.

$a_n = \frac{2}{\pi}\left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$	Right so far: Problem 4.
"$\cos n\pi = 0$, so the bracket is $0 - \frac{1}{n^2}$."	$\sin n\pi = 0$, but $\cos n\pi = (-1)^n$, so the bracket is $\frac{(-1)^n - 1}{n^2}$.
$a_n = -\frac{2}{\pi n^2}$ for every n ✗	Then at $x = 0$ the series gives $\frac{\pi}{2} - \frac{2}{\pi} \cdot \frac{\pi^2}{6} = \frac{\pi}{6} \approx 0.524$ instead of $\lvert 0 \rvert = 0$. The right a_n is $-\frac{4}{\pi n^2}$ for odd n and 0 for even n.

5. The a₀ term left out of Parseval's identity

Parseval is remembered as "the integral of f^2 is the sum of the squared coefficients", for $n \geq 1$.

$\frac{1}{\pi}\int_{-\pi}^{\pi} x^4\,dx = \frac{2\pi^4}{5}$ and $a_n = \frac{4(-1)^n}{n^2}$	Right so far: Problem 8.
"$\frac{2\pi^4}{5} = \sum \frac{16}{n^4}$."	The constant term carries part of the energy: the identity has $\frac{a_0^2}{2} = \frac{2\pi^4}{9}$ on the right as well.
$\sum \frac{1}{n^4} = \frac{\pi^4}{40} \approx 2.435$ ✗	That cannot be right, since $1 + \sum_{n\geq 2} \frac{1}{n^4} < 1 + \sum_{n\geq 2} \frac{1}{n^2} \approx 1.645$. With the $\frac{a_0^2}{2}$ term the sum is $\frac{\pi^4}{90} \approx 1.0823$.

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