Evaluating limits: factoring, conjugates and limits at infinity

Ten limits evaluated exactly: 0/0 forms cleared by factoring, by a difference of cubes and by the conjugate, the difference quotient behind every derivative, sin 3x / x, one-sided limits that disagree, and limits at ±∞ including √(x²) = |x| and ∞ − ∞, with worked solutions and the mistakes that read 0/0 as an answer.

BEFORE YOU START

A limit asks what value $f(x)$ approaches as x approaches a, not what $f(a)$ is, and every derivative is one of them. When substituting a gives a number, that number is the limit; when it gives $\frac{0}{0}$ or $\frac{\infty}{\infty}$, the work is to rewrite f until substitution does give a number. These ten problems run from factoring and the conjugate to the difference quotient, one-sided limits, and limits at infinity, where the sign of $\sqrt{x^2}$ and the form $\infty - \infty$ catch people out.

- $\lim_{x \to a} f(x) = L$ means $f(x)$ gets as close to L as you like for every x close enough to a, with $x \neq a$. The value $f(a)$, or whether $f(a)$ is defined at all, plays no part.
- If f is built from polynomials, roots, exponentials, logs and trig functions by $+$, $-$, $\times$, $\div$ and composition, and $f(a)$ is defined, then $\lim_{x \to a} f(x) = f(a)$: substitute.
- $\frac{0}{0}$ is not a value. It says the limit depends on how fast the top and bottom go to 0, and the expression has to be rewritten before you substitute. Two expressions that agree for every $x \neq a$ have the same limit at a, which is what makes canceling a common factor $(x - a)$ legal.
- One-sided limits: $x \to a^-$ approaches from below, $x \to a^+$ from above. The two-sided limit exists exactly when both one-sided limits exist and are equal.
- Standard limit: $\lim_{t \to 0} \dfrac{\sin t}{t} = 1$, with t in radians.
- At $\pm\infty$, divide the top and bottom by the highest power of x in the denominator, and use $\frac{1}{x^n} \to 0$ for $n > 0$. For a root, $\sqrt{x^2} = |x|$, which is $-x$ when $x < 0$.

Problems

1. Find $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3}$. difficulty ·

2. Find $\lim\limits_{x \to 2} \dfrac{x^2 + x - 6}{x^2 - 4}$. difficulty ·

3. Find $\lim_{x \to 0} \dfrac{\sqrt{x+4} - 2}{x}$. difficulty ··

4. Find $\lim_{x \to 1} \dfrac{x^3 - 1}{x^2 - 1}$. difficulty ··

5. Find $\lim_{h \to 0} \dfrac{(2+h)^2 - 4}{h}$, and say which derivative it is. difficulty ··

6. Find $\lim_{x \to \infty} \dfrac{3x^2 - x + 5}{2x^2 + 7}$. difficulty ··

7. Find $\lim_{x \to -\infty} \dfrac{\sqrt{4x^2 + 1}}{x + 3}$. difficulty ···

8. Find $\lim_{x \to 2^-} \dfrac{|x-2|}{x-2}$ and $\lim_{x \to 2^+} \dfrac{|x-2|}{x-2}$. Does $\lim_{x \to 2} \dfrac{|x-2|}{x-2}$ exist? difficulty ··

9. Find $\lim_{x \to 0} \dfrac{\sin 3x}{x}$. difficulty ···

10. Find $\lim_{x \to \infty} \left(\sqrt{x^2 + 6x} - x \right)$. difficulty ···

Answers

1. 6
2. $\frac{5}{4}$
3. $\frac{1}{4}$
4. $\frac{3}{2}$
5. 4; it is $f'(2)$ for $f(x) = x^2$
6. $\frac{3}{2}$
7. -2
8. -1 from the left, 1 from the right; the two-sided limit does not exist
9. 3
10. 3

Worked solutions

Problem 1

> Find $\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3}$.

Substituting $x = 3$ gives $\frac{9-9}{3-3} = \frac{0}{0}$.	Not an answer: the expression has to be rewritten first.
$\dfrac{x^2 - 9}{x - 3} = \dfrac{(x - 3)(x + 3)}{x - 3} = x + 3$ for $x \neq 3$.	Difference of squares. The factor $x - 3$ is the one that made both the top and the bottom 0, and canceling it is legal because the limit never uses $x = 3$ itself.
$\lim_{x \to 3}(x + 3) = 6$.	$x + 3$ is a polynomial, so substitute.
6	Steps 2 and 3.

Problem 2

> Find $\lim_{x \to 2} \dfrac{x^2 + x - 6}{x^2 - 4}$.

Substituting $x = 2$ gives $\frac{4+2-6}{4-4} = \frac{0}{0}$.	Both polynomials are 0 at 2, so both have the factor $x - 2$.
$x^2 + x - 6 = (x - 2)(x + 3)$ and $x^2 - 4 = (x - 2)(x + 2)$.	Factor both. In the first, -2 and 3 multiply to -6 and add to 1.
$\dfrac{x^2 + x - 6}{x^2 - 4} = \dfrac{x + 3}{x + 2}$ for $x \neq 2$.	Cancel the common factor $x - 2$.
$\frac{5}{4}$	Substitute $x = 2$ into $\frac{x+3}{x+2}$, which is defined there: $\frac{5}{4}$.

Problem 3

> Find $\lim_{x \to 0} \dfrac{\sqrt{x+4}-2}{x}$.

Substituting $x = 0$ gives $\frac{2-2}{0} = \frac{0}{0}$.	There is no polynomial to factor, so use the conjugate of the root expression.
$\dfrac{\sqrt{x+4}-2}{x} \cdot \dfrac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$	Multiplying by $\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$, which is 1, changes the form and not the value. The top and the bottom both get the factor.
$(\sqrt{x+4}-2)(\sqrt{x+4}+2) = (x+4) - 4 = x.$	Difference of squares, $(p-q)(p+q) = p^2 - q^2$: the root squares away.
$\dfrac{x}{x(\sqrt{x+4}+2)} = \dfrac{1}{\sqrt{x+4}+2}$ for $x \neq 0$.	Step 3 on top, the bottom left in factored form so the x can cancel.
$\frac{1}{4}$	Substitute $x = 0$: $\frac{1}{\sqrt{4}+2} = \frac{1}{4}$.

Problem 4

> Find $\lim_{x \to 1} \dfrac{x^3-1}{x^2-1}$.

Substituting $x = 1$ gives $\frac{0}{0}$.	Both are 0 at $x = 1$, so both have the factor $x - 1$.
$x^3 - 1 = (x-1)(x^2+x+1)$ and $x^2 - 1 = (x-1)(x+1)$.	Difference of cubes, $a^3 - b^3 = (a-b)(a^2+ab+b^2)$, and difference of squares.
$\dfrac{x^3-1}{x^2-1} = \dfrac{x^2+x+1}{x+1}$ for $x \neq 1$.	Cancel $x - 1$.
$\frac{3}{2}$	Substitute $x = 1$: $\frac{1+1+1}{1+1} = \frac{3}{2}$.

Problem 5

> Find $\lim_{h \to 0} \dfrac{(2+h)^2 - 4}{h}$, and say which derivative it is.

$(2+h)^2 - 4 = 4 + 4h + h^2 - 4 = 4h + h^2.$	Expand the square, middle term included. Substituting $h = 0$ in the original gives $\frac{0}{0}$, so it has to be rewritten.
$\dfrac{4h + h^2}{h} = 4 + h$ for $h \neq 0.$	Every term on top has a factor h, so it cancels with the bottom.
$\lim_{h \to 0}(4 + h) = 4.$	Substitute $h = 0$.
4; it is $f'(2)$ for $f(x) = x^2$	The derivative is defined as $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, and here $f(a+h) = (2+h)^2$, $f(a) = 4$. It agrees with $(x^2)' = 2x$ at $x = 2$.

Problem 6

> Find $\lim_{x \to \infty} \dfrac{3x^2 - x + 5}{2x^2 + 7}$.

Substituting is not possible, and the top and bottom both grow without bound: the form is $\frac{\infty}{\infty}$.	Rewrite so that every term has a limit.
$\dfrac{3x^2 - x + 5}{2x^2 + 7} = \dfrac{3 - \frac{1}{x} + \frac{5}{x^2}}{2 + \frac{7}{x^2}}$	Divide every term, top and bottom, by x^2, the highest power in the denominator. This is multiplying by $\frac{1/x^2}{1/x^2} = 1$.
$\frac{1}{x} \to 0$, $\frac{5}{x^2} \to 0$ and $\frac{7}{x^2} \to 0$ as $x \to \infty$.	A fixed number divided by an ever larger one.
$\frac{3}{2}$	The limit of the top is 3 and of the bottom is 2. When the top and bottom have the same degree, the limit is the ratio of the leading coefficients.

Problem 7

> Find $\lim_{x \to -\infty} \dfrac{\sqrt{4x^2+1}}{x+3}$.

$\sqrt{4x^2+1} = \sqrt{x^2}\sqrt{4+\frac{1}{x^2}} = \lvert x\rvert\sqrt{4+\frac{1}{x^2}}.$	Factor x^2 out from under the root. $\sqrt{x^2}$ is $\lvert x\rvert$, not x: the root is never negative.
$\lvert x\rvert = -x$ for $x < 0$.	$x \to -\infty$, so from some point on x is negative.
$\dfrac{\sqrt{4x^2+1}}{x+3} = \dfrac{-x\sqrt{4+\frac{1}{x^2}}}{x\left(1+\frac{3}{x}\right)} = \dfrac{-\sqrt{4+\frac{1}{x^2}}}{1+\frac{3}{x}}$	Steps 1 and 2 on top, x factored out of the bottom, then cancel x.
-2	$\frac{1}{x^2} \to 0$ and $\frac{3}{x} \to 0$, so the limit is $\frac{-\sqrt{4}}{1} = -2$. Sanity check: the top is positive and the bottom is negative for large negative x, so the limit has to be negative.

Problem 8

> Find $\lim_{x \to 2^-} \dfrac{\lvert x-2\rvert}{x-2}$ and $\lim_{x \to 2^+} \dfrac{\lvert x-2\rvert}{x-2}$. Does $\lim_{x \to 2} \dfrac{\lvert x-2\rvert}{x-2}$ exist?

For $x < 2$, $\lvert x-2\rvert = -(x-2)$, so $\dfrac{\lvert x-2\rvert}{x-2} = -1$.	Below 2, $x-2$ is negative and the absolute value flips its sign.
$\lim_{x \to 2^-} \dfrac{\lvert x-2\rvert}{x-2} = -1.$	The expression is the constant -1 for every x just below 2.
For $x > 2$, $\lvert x-2\rvert = x-2$, so the expression is 1 and $\lim_{x \to 2^+} = 1$.	Above 2, $x-2$ is positive and the absolute value changes nothing.
-1 from the left, 1 from the right; the two-sided limit does not exist	The one-sided limits exist but differ, so there is no single value that $f(x)$ approaches from both sides.

Problem 9

Find $\lim_{x \to 0} \dfrac{\sin 3x}{x}$.

Substituting $x = 0$ gives $\frac{\sin 0}{0} = \frac{0}{0}$.	The standard limit $\frac{\sin t}{t} \to 1$ applies only when the angle and the denominator are the same quantity.
$\dfrac{\sin 3x}{x} = 3 \cdot \dfrac{\sin 3x}{3x}$	Multiply the top and the bottom by 3, so the denominator matches the angle.
$t = 3x \to 0$ as $x \to 0$, so $\dfrac{\sin 3x}{3x} = \dfrac{\sin t}{t} \to 1$.	The substitution $t = 3x$ turns the expression into the standard limit.
3	$3 \cdot 1$. The same argument gives $\lim_{x \to 0} \frac{\sin kx}{x} = k$ for any constant k.

Problem 10

Find $\lim_{x \to \infty} \left(\sqrt{x^2 + 6x} - x \right)$.

Both $\sqrt{x^2 + 6x}$ and x go to ∞: the form is $\infty - \infty$.	Like $\frac{0}{0}$, that is not a value. Multiply by the conjugate to turn the difference into a quotient.
$\left(\sqrt{x^2 + 6x} - x \right) \dfrac{\sqrt{x^2 + 6x} + x}{\sqrt{x^2 + 6x} + x} = \dfrac{(x^2 + 6x) - x^2}{\sqrt{x^2 + 6x} + x}$	Difference of squares on top: the x^2 terms cancel.
$\dfrac{6x}{\sqrt{x^2 + 6x} + x} = \dfrac{6x}{x\sqrt{1 + \frac{6}{x}} + x} = \dfrac{6}{\sqrt{1 + \frac{6}{x}} + 1}$	$\sqrt{x^2 + 6x} = \lvert x \rvert \sqrt{1 + \frac{6}{x}}$ and $\lvert x \rvert = x$ for $x > 0$; then cancel x.
3	$\frac{6}{x} \to 0$, so the limit is $\frac{6}{1+1} = 3$. The root is about $x + 3$ for large x, since $(x + 3)^2 = x^2 + 6x + 9$.

Where this goes wrong

1. Reading 0/0 as the answer

Substitution is the first thing to try, and when it produces $\frac{0}{0}$ it looks like a result.

$\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3}$: substituting gives $\frac{0}{0}$	Right so far: Problem 1, step 1.
"Division by zero is undefined, so the limit is undefined."	The habit that causes the mistake: treating the limit as the value at $x = 3$.
$\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3}$ does not exist ✗	The function is undefined at 3, but the limit never uses $x = 3$. For every other x it equals $x + 3$, which approaches 6; at $x = 3.001$ the expression is 6.001. $\frac{0}{0}$ is a signal to rewrite, never a final answer, and neither is "0" or "1".

2. Canceling terms instead of factors

The x^2 on top and the x^2 on the bottom look like they should cancel.

$\lim_{x \to 2} \dfrac{x^2 + x - 6}{x^2 - 4}$	Right so far: Problem 2.
"Cancel the x^2 from the top and the bottom."	The habit that causes the mistake: canceling a term that is added, not multiplied.
$\dfrac{x^2 + x - 6}{x^2 - 4} = \dfrac{x - 6}{-4}$, so the limit is $\frac{-4}{-4} = 1$ ✗	Only a factor of the whole top and the whole bottom cancels. At $x = 3$ the left side is $\frac{6}{5}$ and the right side is $\frac{3}{4}$, so the two are not equal. Factor first: the common factor is $x - 2$, and the limit is $\frac{5}{4}$ (Problem 2).

3. Multiplying only the top by the conjugate

The conjugate is there to remove the root from the top, so the bottom gets left alone.

$\lim_{x\to 0} \dfrac{\sqrt{x+4}-2}{x}$, using the conjugate $\sqrt{x+4}+2$	Right so far: Problem 3.
"$(\sqrt{x+4}-2)(\sqrt{x+4}+2) = x$, so the numerator becomes x."	The numerator is right; the bottom has not been touched.
$\dfrac{\sqrt{x+4}-2}{x} = \dfrac{x}{x} = 1$ ✗	Multiplying only the top by $\sqrt{x+4}+2$ multiplies the whole expression by a number near 4, which changes it. The bottom must become $x(\sqrt{x+4}+2)$; the answer is $\frac{1}{4}$, not 1.

4. Writing √(x²) = x when x → −∞

Pulling x out of a square root is a reflex, and for positive x it is right.

$\lim_{x\to -\infty} \dfrac{\sqrt{4x^2+1}}{x+3}$	Right so far: Problem 7.
"$\sqrt{4x^2+1} = x\sqrt{4+1/x^2}$."	The habit that causes the mistake: $\sqrt{x^2} = x$, which holds only for $x \geq 0$.
$\dfrac{x\sqrt{4+1/x^2}}{x(1+3/x)} \to 2$ ✗	$\sqrt{x^2} = \lvert x\rvert = -x$ for negative x, so a minus sign is missing. At $x = -1000$ the expression is about $\frac{2000}{-997} \approx -2.006$: a positive root over a negative denominator. The limit is -2.

5. Treating ∞ − ∞ as 0

For large x, $\sqrt{x^2+6x}$ and x are both huge and nearly equal, so their difference looks negligible.

$\lim_{x\to\infty} \left(\sqrt{x^2+6x} - x\right)$	Right so far: Problem 10.
"For large x, $x^2+6x \approx x^2$, so $\sqrt{x^2+6x} \approx x$."	True as a ratio, $\sqrt{x^2+6x}/x \to 1$, but a ratio near 1 says nothing about the difference.

$$\lim_{x\to\infty}\left(\sqrt{x^2+6x}-x\right)=\infty-\infty=0\ \text{✗}$$

At $x=10^6$ the difference is $2.999995\ldots$, nowhere near 0. The $6x$ that was dropped is exactly what the difference measures. Use the conjugate (Problem 10): the limit is 3.

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