Eigenvalues and eigenvectors by hand

Ten problems on eigenvalues and eigenvectors worked by hand: 2×2 and 3×3 matrices, the characteristic polynomial, a rotation with complex eigenvalues, a shear with too few eigenvectors, diagonalization and powers, orthogonal eigenvectors of symmetric matrices, power iteration and singular values, with worked solutions and the classic sign and null-space mistakes.

BEFORE YOU START

Every eigenvalue problem on this page comes down to two computations: a determinant that gives the eigenvalues, and a small linear system for each one that gives the eigenvectors. The errors come from the same few places: a sign in $A - \lambda I$, an eigenvector read off a row instead of solved for, a repeated eigenvalue assumed to bring two eigenvectors. These ten problems start with 2×2 matrices, meet a rotation and a shear that break the easy pattern, and end with the three uses machine learning makes of eigenvectors: powers of a matrix, power iteration and singular values.

- A is a real $n \times n$ matrix and I the $n \times n$ identity. A number λ is an eigenvalue of A, with eigenvector v, when $Av = \lambda v$ and $v \neq 0$.
- $Av = \lambda v$ is the same as $(A - \lambda I)v = 0$. That has a nonzero solution exactly when $A - \lambda I$ is singular, that is when $\det(A - \lambda I) = 0$. $\det(A - \lambda I)$ is a polynomial of degree n in λ, the characteristic polynomial, and its roots are the eigenvalues.
- An eigenvector is any nonzero solution of $(A - \lambda I)v = 0$, so the answers below give one representative each; any nonzero multiple is also correct.
- Vectors are columns; $(1, -1)^\top$ is the column with entries 1 and -1, written on one line to save space.
- For 2×2 matrices: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$, and when that is not 0, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. $\operatorname{tr} A$, the trace, is the sum of the diagonal entries.
- P is a matrix whose columns are eigenvectors of A. D (Problem 7) and Λ (Problem 8) are both diagonal matrices of eigenvalues, listed in the same order as the columns of P; $\operatorname{diag}(a, b)$ is the diagonal matrix with a and b on the diagonal. Λ is used when the eigenvectors are orthonormal, and then the eigenvector matrix is written Q: an orthogonal matrix, meaning $Q^\top Q = I$, so $Q^{-1} = Q^\top$.
- i is the imaginary unit, $i^2 = -1$. σ_1, σ_2 are singular values (Problem 10).

Problems

1. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. difficulty ·

2. Find the characteristic polynomial, eigenvalues and eigenvectors of $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$. difficulty ·

3. Find the eigenvalues of the rotation $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Does it have a real eigenvector? What does that mean geometrically? difficulty ··

4. Find the eigenvalues and eigenvectors of the shear $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Can it be diagonalized? difficulty ··

5. For any 2×2 matrix, show that the trace is the sum of the eigenvalues and the determinant is their product. Verify on Problem 2. difficulty ··

6. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. difficulty ···

7. Diagonalize the matrix of Problem 2 as $A = PDP^{-1}$ and use it to compute A^5. difficulty ··

8. Show that the eigenvectors of Problem 1 are orthogonal, and write $A = Q\Lambda Q^{\top}$ with Q orthogonal. Why does this always happen for symmetric matrices? difficulty ··

9. Apply $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ repeatedly to $x_0 = (1, 0)^{\top}$. Find $A^k x_0$ in closed form and say what direction it approaches and how fast. difficulty ···

10. For $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$, find the eigenvalues of $A^{\top}A$ and hence the singular values of A. Check them against $\det A$. difficulty ···

Answers

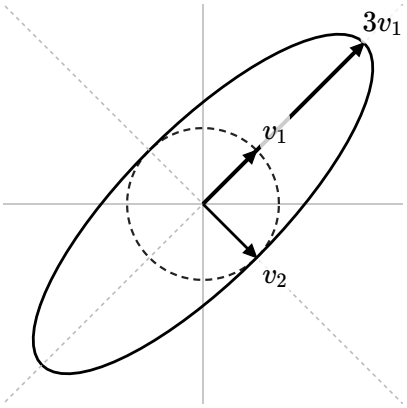
1. $\lambda = 3$ with $v = (1, 1)^\top$; $\lambda = 1$ with $v = (1, -1)^\top$
2. $\lambda^2 - 7\lambda + 10 = (\lambda - 5)(\lambda - 2)$; $\lambda = 5$, $v = (1, 1)^\top$; $\lambda = 2$, $v = (1, -2)^\top$
3. $\lambda^2 + 1 = 0$, $\lambda = \pm i$; no real eigenvector: a rotation by 90° sends every real direction somewhere else
4. $(\lambda - 1)^2$, $\lambda = 1$ (double); $A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has rank 1, so the only eigenvectors are multiples of $(1, 0)^\top$; not diagonalizable
5. $\operatorname{tr} A = \lambda_1 + \lambda_2$, $\det A = \lambda_1 \lambda_2$; Problem 2: $7 = 5 + 2$, $10 = 5 \cdot 2$
6. $\lambda = 2$ with $(1, 0, -1)^\top$; $\lambda = 2 \pm \sqrt{2}$ with $(1, \pm\sqrt{2}, 1)^\top$
7. $P = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$, $D = \operatorname{diag}(5, 2)$, $P^{-1} = -\frac{1}{3}\begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix}$; $A^5 = PD^5P^{-1} = \begin{bmatrix} 2094 & 1031 \\ 2062 & 1063 \end{bmatrix}$
8. $Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\Lambda = \operatorname{diag}(3, 1)$, $A = Q\Lambda Q^\top$; for symmetric A, $\lambda_1 v_1^\top v_2 = (Av_1)^\top v_2 = v_1^\top A v_2 = \lambda_2 v_1^\top v_2$, so distinct eigenvalues force $v_1^\top v_2 = 0$
9. $A^k x_0 = \frac{3^k}{2}(1, 1)^\top + \frac{1}{2}(1, -1)^\top$; the direction approaches $(1, 1)/\sqrt{2}$, the error shrinking like $(1/3)^k$ — the ratio of the two eigenvalues
10. $A^\top A = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}$ with eigenvalues 45 and 5; singular values $3\sqrt{5}$ and $\sqrt{5}$; their product $15 = |\det A|$

Worked solutions

Problem 1

> Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

$A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$, so $\det(A - \lambda I) = (2-\lambda)^2 - 1$.	Subtract λ from each diagonal entry, then $ad - bc$.
$(2-\lambda)^2 = 1$, so $2 - \lambda = \pm 1$ and $\lambda = 1$ or $\lambda = 3$.	The eigenvalues are the roots of the characteristic polynomial; this one is a square minus 1, so take square roots rather than expanding.
$\lambda = 3$: $A - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, and with $v = (x, y)^\top$, $(A - 3I)v = 0$ says $-x + y = 0$, so $v = (1, 1)^\top$.	The eigenvectors for $\lambda = 3$ are the null space of $A - 3I$. Its two rows are multiples of each other, as they must be when the determinant is 0, so one equation remains; choose $x = 1$.
$\lambda = 1$: $A - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, and $(A - I)v = 0$ says $x + y = 0$, so $v = (1, -1)^\top$.	Null space of $A - I$; the equation $x + y = 0$ fixes v up to scale.
$\lambda = 3$ **with** $v = (1, 1)^\top$; $\lambda = 1$ **with** $v = (1, -1)^\top$	Check: $A(1, 1)^\top = (3, 3)^\top$ and $A(1, -1)^\top = (1, -1)^\top$.



Problem 2

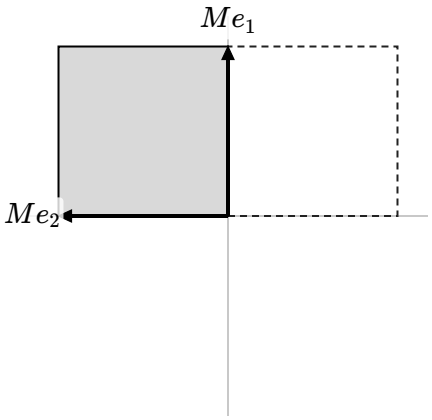
> Find the characteristic polynomial, eigenvalues and eigenvectors of $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$.

$\det(A - \lambda I) = (4 - \lambda)(3 - \lambda) - 1 \cdot 2.$	$ad - bc$ for $A - \lambda I = \begin{bmatrix} 4 - \lambda & 1 \\ 2 & 3 - \lambda \end{bmatrix}$.
$(4 - \lambda)(3 - \lambda) - 2 = \lambda^2 - 7\lambda + 12 - 2 = \lambda^2 - 7\lambda + 10.$	Expand the product and collect terms.
$\lambda^2 - 7\lambda + 10 = (\lambda - 5)(\lambda - 2).$	Two numbers with product 10 and sum 7.
$\lambda = 5$: $A - 5I = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$ gives $-x + y = 0$ for $v = (x, y)^\top$, so $v = (1, 1)^\top$.	Null space of $A - 5I$; the second row is -2 times the first, so it adds nothing.
$\lambda = 2$: $A - 2I = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$ gives $2x + y = 0$, so $v = (1, -2)^\top$.	Null space of $A - 2I$, with $x = 1$.
$\lambda^2 - 7\lambda + 10 = (\lambda - 5)(\lambda - 2)$; $\lambda = 5$, $v = (1, 1)^\top$; $\lambda = 2$, $v = (1, -2)^\top$	Check: $A(1, 1)^\top = (5, 5)^\top$ and $A(1, -2)^\top = (2, -4)^\top$. The eigenvectors are not perpendicular; A is not symmetric, and Problem 8 shows why that matters.

Problem 3

> Find the eigenvalues of the rotation $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Does it have a real eigenvector? What does that mean geometrically?

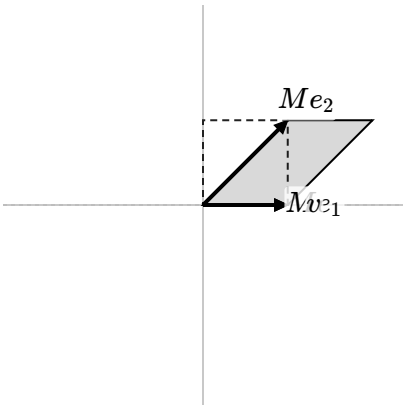
$\det(R - \lambda I) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1.$	$ad - bc = (-\lambda)(-\lambda) - (-1)(1).$
$\lambda^2 + 1 = 0$ has no real root; $\lambda = \pm i$.	$\lambda^2 \geq 0$ for every real λ, so $\lambda^2 + 1 \geq 1$; the two roots are complex.

No real eigenvector.	A real eigenvector v would make $Rv = \lambda v$ with both sides real, so λ would be real; there is no real eigenvalue. The eigenvectors are complex: $\lambda = i$ has $v = (1, -i)^\top$, since $R(1, -i)^\top = (i, 1)^\top = i(1, -i)^\top$.
$R(x, y)^\top = (-y, x)^\top$, which is $v = (x, y)^\top$ turned 90° counterclockwise. 	$v \cdot Rv = -xy + yx = 0$ and $\|Rv\| = \|v\|$, so for every nonzero real v, Rv is a nonzero vector perpendicular to v, off the line through v. An eigenvector is exactly a vector that stays on its own line.
$\lambda^2 + 1 = 0$, $\lambda = \pm i$; **no real eigenvector: a rotation by 90° sends every real direction somewhere else**	Steps 2 to 4.

Problem 4

> Find the eigenvalues and eigenvectors of the shear $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Can it be diagonalized?

$\det(A - \lambda I) = (1 - \lambda)^2 - 1 \cdot 0 = (\lambda - 1)^2.$	$A - \lambda I$ is upper triangular, so its determinant is the product of the diagonal entries.

$\lambda = 1$, a double root.	The only root, counted twice: its algebraic multiplicity, the number of times it is a root, is 2.
$A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has one nonzero row, so $\operatorname{rank}(A - I) = 1$.	The rank decides how many independent eigenvectors $\lambda = 1$ has: the null space of a 2×2 matrix has dimension $2 - \operatorname{rank}$.
For $v = (x, y)^\top$, $(A - I)v = 0$ says $y = 0$, so every eigenvector is $(x, 0)^\top$, a multiple of $(1, 0)^\top$. 	The null space has dimension $2 - 1 = 1$: the geometric multiplicity, the number of independent eigenvectors, is 1.
Not diagonalizable.	$A = PDP^{-1}$ needs two independent eigenvectors for the columns of an invertible P, and there is only one direction. Directly: D would have to be $\operatorname{diag}(1, 1) = I$, and then $PIP^{-1} = I \neq A$.
$(\lambda - 1)^2$, $\lambda = 1$ **(double);** $A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ **has rank** 1**, so the only eigenvectors are multiples of** $(1, 0)^\top$**; not diagonalizable**	Steps 1 to 5.

Problem 5

> For any 2×2 matrix, show that the trace is the sum of the eigenvalues and the determinant is their product. Verify on Problem 2.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\det(A - \lambda I) = (a-\lambda)(d-\lambda) - bc$.	$ad - bc$ for $A - \lambda I$.
$(a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad - bc)$.	Expand and collect powers of λ.
$\det(A - \lambda I) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$.	λ_1 and λ_2 are the two roots, complex or repeated if need be, and a quadratic whose λ^2 coefficient is 1 is the product of $(\lambda - \text{root})$ over its roots.
$a + d = \lambda_1 + \lambda_2$ and $ad - bc = \lambda_1\lambda_2$.	Steps 2 and 3 are the same polynomial, so their coefficients of λ and their constant terms agree.
$\operatorname{tr} A = \lambda_1 + \lambda_2$, $\det A = \lambda_1\lambda_2$; **Problem 2:** $7 = 5 + 2$, $10 = 5 \cdot 2$	For Problem 2, $\operatorname{tr} A = 4 + 3 = 7$ and $\det A = 4 \cdot 3 - 1 \cdot 2 = 10$. This is a quick check on any eigenvalue computation.

Problem 6

> Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

Write $t = 2 - \lambda$, so the diagonal entries of $A - \lambda I$ are all t and the off-diagonal entries are those of A. Each eigenvector is written $v = (x, y, z)^\top$.

$\det(A - \lambda I) = t \det \begin{bmatrix} t & 1 \\ 1 & t \end{bmatrix} - 1 \cdot \det \begin{bmatrix} 1 & 1 \\ 0 & t \end{bmatrix} + 0$.	Cofactor expansion along the first row, $(t, 1, 0)$: each entry times the determinant left after deleting its row and column, with signs $+, -, +$.
$t(t^2 - 1) - t = t^3 - 2t = t(t^2 - 2)$.	The two 2×2 determinants are $t^2 - 1$ and $t - 0$.

$\det(A-\lambda I) = (2-\lambda)\big((2-\lambda)^2-2\big).$	Substitute $t = 2-\lambda$ back.
$\lambda = 2$, or $2-\lambda = \pm\sqrt{2}$, that is $\lambda = 2 \mp \sqrt{2}$.	Set each factor to 0. Check with the trace: $2 + (2-\sqrt{2}) + (2+\sqrt{2}) = 6 = \operatorname{tr} A$.
$\lambda = 2$ ($t = 0$): the rows of $A - 2I$ give $y = 0$, $x + z = 0$, $y = 0$, so $v = (1, 0, -1)^\top$.	The null space of $A - 2I$, whose rows are $(0,1,0)$, $(1,0,1)$, $(0,1,0)$.
$\lambda = 2 + \sqrt{2}$ ($t = -\sqrt{2}$): the rows give $-\sqrt{2}\,x + y = 0$, $x - \sqrt{2}\,y + z = 0$, $y - \sqrt{2}\,z = 0$.	The null space of $A - (2+\sqrt{2})I$, whose diagonal entries are $-\sqrt{2}$.
$y = \sqrt{2}\,x$ and $y = \sqrt{2}\,z$, so $z = x$; with $x = 1$, $v = (1, \sqrt{2}, 1)^\top$.	The first and third equations; the middle one then holds, $1 - 2 + 1 = 0$, as it must because the determinant is 0.
$\lambda = 2 - \sqrt{2}$ ($t = \sqrt{2}$): the same equations with $\sqrt{2}$ in place of $-\sqrt{2}$ give $y = -\sqrt{2}\,x = -\sqrt{2}\,z$, so $v = (1, -\sqrt{2}, 1)^\top$.	Only the sign of t changed, so only the sign of y changes.
$\lambda = 2$ **with** $(1,0,-1)^\top$; $\lambda = 2 \pm \sqrt{2}$ **with** $(1, \pm\sqrt{2}, 1)^\top$	Check: $A(1,\sqrt{2},1)^\top = (2+\sqrt{2},\ 2+2\sqrt{2},\ 2+\sqrt{2})^\top = (2+\sqrt{2})(1,\sqrt{2},1)^\top$. A is symmetric, and the three eigenvectors are mutually perpendicular (Problem 8): divided by their lengths, they are the columns of an orthogonal Q.

$$\underbrace{\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \end{bmatrix}}_{Q^\top} \cdot \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \end{bmatrix}}_{Q} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$$

Problem 7

> Diagonalize the matrix of Problem 2 as $A = PDP^{-1}$ and use it to compute A^5.

$P = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$, $D = \operatorname{diag}(5, 2)$.	The columns of P are the eigenvectors from Problem 2, and D lists their eigenvalues in the same order. Then $AP = PD$ column by column, which is $Av = \lambda v$ twice.
$\det P = 1 \cdot (-2) - 1 \cdot 1 = -3$, so $P^{-1} = -\frac{1}{3}\begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix}$.	The 2×2 inverse: swap the diagonal, negate the off-diagonal, divide by the determinant. $\det P \neq 0$ because the two eigenvectors are independent.
$PD = \begin{bmatrix} 5 & 2 \\ 5 & -4 \end{bmatrix}$ and $PDP^{-1} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} = A$.	Check the diagonalization by multiplying it out: the first row of PDP^{-1} is $(10/3 + 2/3,\ 5/3 - 2/3) = (4, 1)$.
$A^5 = PDP^{-1}\,PDP^{-1} \cdots PDP^{-1} = PD^5P^{-1}$.	Each inner $P^{-1}P$ is I, so the five copies collapse. Powers of a diagonal matrix are powers of its entries: $D^5 = \operatorname{diag}(3125, 32)$.
$PD^5 = \begin{bmatrix} 3125 & 32 \\ 3125 & -64 \end{bmatrix}$.	Multiplying by a diagonal matrix on the right scales the columns of P by 5^5 and 2^5.
$\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3125 & 0 \\ 0 & 32 \end{bmatrix} = \begin{bmatrix} 3125 & 32 \\ 3125 & -64 \end{bmatrix}$ P $\quad$ D^5 $\quad$ PD^5	
$PD^5P^{-1} = \frac{1}{3}\begin{bmatrix} 6250 + 32 & 3125 - 32 \\ 6250 - 64 & 3125 + 64 \end{bmatrix} = \begin{bmatrix} 2094 & 1031 \\ 2062 & 1063 \end{bmatrix}$.	Row times column with $P^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$.

$P = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$, $D = \operatorname{diag}(5, 2)$, $P^{-1} = -\frac{1}{3}\begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix}$; $A^5 = PD^5P^{-1} = \begin{bmatrix} 2094 & 1031 \\ 2062 & 1063 \end{bmatrix}$	Check: $(1,1)^\top$ is an eigenvector with eigenvalue 5, so $A^5(1,1)^\top = 3125(1,1)^\top$, and both rows do sum to 3125.

Problem 8

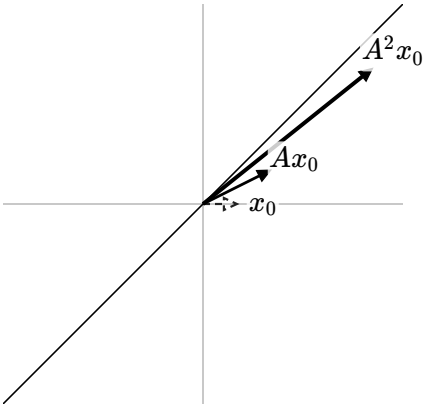
> Show that the eigenvectors of Problem 1 are orthogonal, and write $A = Q\Lambda Q^\top$ with Q orthogonal. Why does this always happen for symmetric matrices?

$(1,1)^\top \cdot (1,-1)^\top = 1 - 1 = 0.$	Orthogonal means a zero dot product.
$Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\Lambda = \operatorname{diag}(3, 1).$	Divide each eigenvector by its length $\sqrt{2}$ so the columns are unit vectors; they are still eigenvectors. Λ lists the eigenvalues in column order.
$Q^\top Q = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I.$	The columns are orthogonal unit vectors, so Q is orthogonal and $Q^{-1} = Q^\top$.
$Q\Lambda Q^\top = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = A.$	$A = Q\Lambda Q^{-1}$ as in Problem 7, with Q^{-1} replaced by $Q^\top$; multiplying it out confirms it.
For symmetric A with $Av_1 = \lambda_1 v_1$, $Av_2 = \lambda_2 v_2$: $\lambda_1 v_1^\top v_2 = (Av_1)^\top v_2 = v_1^\top A^\top v_2 = v_1^\top A v_2 = \lambda_2 v_1^\top v_2$.	Move A across the dot product: $(Av_1)^\top = v_1^\top A^\top$, and $A^\top = A$ is where symmetry is used.
$(\lambda_1 - \lambda_2)\, v_1^\top v_2 = 0$, so $v_1^\top v_2 = 0$ when $\lambda_1 \neq \lambda_2$.	Subtract the two ends of step 5; a product is 0 only if a factor is. A repeated eigenvalue of a symmetric matrix has as many independent eigenvectors as its multiplicity (the spectral theorem), and they can be chosen orthogonal, so $A = Q\Lambda Q^\top$ holds for every real symmetric matrix.

$Q = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$, $\Lambda = \operatorname{diag}(3, 1)$, $A = Q\Lambda Q^\top$; **for symmetric** A, $\lambda_1 v_1^\top v_2 = (Av_1)^\top v_2 = v_1^\top A v_2 = \lambda_2 v_1^\top v_2$, **so distinct eigenvalues force** $v_1^\top v_2 = 0$	Steps 1 to 6.

Problem 9

Apply $A = \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$ repeatedly to $x_0 = (1, 0)^\top$. Find $A^k x_0$ in closed form and say what direction it approaches and how fast.

$x_0 = \frac{1}{2}(1, 1)^\top + \frac{1}{2}(1, -1)^\top$.	Write x_0 in the eigenvectors from Problem 1, where A acts by scaling; the coefficients solve $c_1 + c_2 = 1$, $c_1 - c_2 = 0$.
$A^k(1, 1)^\top = 3^k(1, 1)^\top$ and $A^k(1, -1)^\top = 1^k(1, -1)^\top = (1, -1)^\top$.	Each application of A multiplies an eigenvector by its eigenvalue, so k applications multiply it by the k-th power.
$A^k x_0 = \frac{3^k}{2}(1, 1)^\top + \frac{1}{2}(1, -1)^\top$. 	A^k is linear, so apply step 2 to each term of step 1. Check: $k = 1$ gives $(2, 1)^\top = Ax_0$, and $k = 2$ gives $(5, 4)^\top$.
$A^k x_0 = \frac{3^k}{2}\big((1, 1)^\top + 3^{-k}(1, -1)^\top\big)$.	Factor out the growing term; the positive scale factor does not change the direction.

$\tan\theta_k = 3^{-k}$, where θ_k is the angle between $A^k x_0$ and $(1,1)^\top$.	In step 4 the two eigenvectors are perpendicular and have the same length $\sqrt{2}$, so the ratio of their coefficients, 3^{-k}, is the tangent of that angle, and it goes to 0.
$A^k x_0 = \frac{3^k}{2}(1,1)^\top + \frac{1}{2}(1,-1)^\top$; **the direction approaches** $(1,1)/\sqrt{2}$**, the error shrinking like** $(1/3)^k$ **— the ratio of the two eigenvalues**	This is power iteration: repeated multiplication (with a rescaling each step to keep the length finite, since it grows like 3^k) finds the eigenvector of the largest eigenvalue, at a rate set by $\lambda_2/\lambda_1 = 1/3$.

Problem 10

> For $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$, find the eigenvalues of $A^\top A$ and hence the singular values of A. Check them against $\det A$.

$A^\top A = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 9+16 & 20 \\ 20 & 25 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}$.	Row times column; the result is symmetric, as $A^\top A$ always is.
$\det(A^\top A - \lambda I) = (25-\lambda)^2 - 400$, so $25 - \lambda = \pm 20$ and $\lambda = 45$ or 5.	The same shape as Problem 1: a square minus a constant. The eigenvectors are $(1,1)^\top$ and $(1,-1)^\top$ again.
$\sigma_1 = \sqrt{45} = 3\sqrt{5}$ and $\sigma_2 = \sqrt{5}$.	The singular values are the square roots of the eigenvalues of $A^\top A$: if $A = U\Sigma V^\top$ with U, V orthogonal and $\Sigma = \operatorname{diag}(\sigma_1, \sigma_2)$, then $A^\top A = V\Sigma^2 V^\top$, an orthogonal diagonalization as in Problem 8 with $\Lambda = \Sigma^2$. The eigenvalues are never negative, since $v^\top A^\top A v = \|Av\|^2 \geq 0$.
$\sigma_1\sigma_2 = 3\sqrt{5}\cdot\sqrt{5} = 15$, and $\det A = 3\cdot 5 - 0\cdot 4 = 15$.	$\det(A^\top A) = (\det A)^2$ is the product of its eigenvalues $\sigma_1^2\sigma_2^2$ (Problem 5), so $\sigma_1\sigma_2 = \lvert\det A\rvert$; here $45\cdot 5 = 225 = 15^2$.

$A^\top A = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}$ **with eigenvalues** 45 **and** 5; **singular values** $3\sqrt{5}$ **and** $\sqrt{5}$; **their product** $15 = \lvert \det A \rvert$	The eigenvalues of A itself are 3 and 5, its diagonal entries since it is triangular: same product, but not the singular values. They coincide, up to sign, only for special matrices such as symmetric ones.

Where this goes wrong

1. Wrong sign in the characteristic matrix

$Av = \lambda v$ has λ on the right, and moving it across is where the sign slips.

$Av = \lambda v$ for $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$	Right so far: the eigen-equation of Problem 1.
"Collect everything on one side: A and λI together."	The step that causes the mistake: the two terms are gathered without tracking the sign.
$\det(A + \lambda I) = (2 + \lambda)^2 - 1 = 0$, so $\lambda = -1$ or $\lambda = -3$ ✗	$Av = \lambda v$ rearranges to $Av - \lambda v = 0$, that is $(A - \lambda I)v = 0$. With the plus sign every root comes out negated: -1 and -3 instead of 1 and 3. The trace catches it: the eigenvalues must sum to $\operatorname{tr} A = 4$ (Problem 5), and these sum to -4.

2. Reading the eigenvector off a row of A − λI

The rows of $A - \lambda I$ are the first vectors on the page after the eigenvalue is found, and it is tempting to take one as the answer.

$\lambda = 3$ in Problem 1, and $A - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$	Right so far.
"The eigenvector comes from $A - 3I$, so read it off the first row."	The step that causes the mistake: the row is taken as the vector instead of as an equation the vector must satisfy.

$v = (-1, 1)^\top$ ✗	The eigenvector is in the *null space* of $A - 3I$, which is orthogonal to its rows: the row $(-1, 1)$ says $-x + y = 0$, and the solution $(1, 1)^\top$ is perpendicular to it. Here the wrong vector differs from the right one only in the sign of one entry, which is why the habit survives, but $A(-1, 1)^\top = (-1, 1)^\top$: it is the eigenvector for $\lambda = 1$, not 3. For a 2×2 row (a, b), the correct move is to swap the entries and negate one, giving $(b, -a)$.

3. Assuming a double eigenvalue gives two eigenvectors

Two eigenvalues usually bring two eigenvectors, and a double eigenvalue looks like two eigenvalues.

The shear $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has $\det(A - \lambda I) = (\lambda - 1)^2$	Right so far: Problem 4.
"A 2×2 matrix has two eigenvalues, so it has two eigenvectors."	The count that causes the mistake: it counts roots, not independent solutions.
"$\lambda = 1$ twice, so $\mathbb{R}^2$ is spanned by eigenvectors and $A = PDP^{-1}$." ✗	The algebraic multiplicity is 2 but the geometric multiplicity is 1: $A - I$ has rank 1, so its null space holds only multiples of $(1, 0)^\top$ (Problem 4). The rank computation is the test; without it the claim would force $A = PIP^{-1} = I$.

4. P and P⁻¹ swapped

P and P^{-1} appear on opposite sides of D, and which side is which is easy to forget.

$P = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$, $D = \operatorname{diag}(5, 2)$ for the matrix of Problem 2	Right so far: Problem 7.

"A is D with P on one side and P^{-1} on the other."	The half-memory that causes the mistake: it does not say which side.
$A^5 = P^{-1}D^5P$ ✗	$A = PDP^{-1}$ because $AP = PD$: P holds the eigenvectors as columns and multiplies on the left. The swapped product is not A^5; here it is $\begin{bmatrix}2094 & 2062\\1031 & 1063\end{bmatrix}$, the transpose of the right answer — only because this P happens to be symmetric; in general the swapped product is just a different matrix. The eigenvector check catches it: $A^5(1,1)^\top$ must be $3125(1,1)^\top$, but the first row of this matrix sums to 4156.

5. Complex eigenvalues read as no eigenvalues

In school algebra a negative discriminant means "no solution", and the habit carries over.

The rotation $R = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$ has $\det(R - \lambda I) = \lambda^2 + 1$	Right so far: Problem 3.
"$\lambda^2 + 1 = 0$ has no solution."	The habit that causes the mistake: no *real* solution is read as no solution.
"The rotation has no eigenvalues." ✗	The characteristic polynomial of an $n \times n$ matrix always has n roots over $\mathbb{C}$, counted with multiplicity; here they are $\pm i$. What the rotation lacks is a *real* eigenvector, because no real direction stays on its own line under a 90° turn.

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