Double integrals: iterated integrals, switching the order and polar coordinates

Ten double integrals worked by hand: iterated integrals over a rectangle in both orders, a product that splits into two single integrals, a triangle and the region between y = x² and y = x, switching the order to integrate e^(y²) and sin(y)/y, polar coordinates over a disk and a quarter disk, the Gaussian integral √π, and a joint density with a probability, two expectations and a covariance, with worked solutions and the mistakes of swapping limits without redrawing the region, dropping the r in r dr dθ and squaring an integral with one variable.

BEFORE YOU START

A double integral adds up $f(x, y)$ over a region of the plane: a volume under a surface, a total mass, or, when f is a joint density, a probability. It is computed as two single integrals, one inside the other, and nearly all the work is in the limits. The inner limits may depend on the outer variable, never the other way round; switching the order means redrawing the region and describing it again, not swapping symbols; and polar coordinates bring an extra factor r with the area element. These ten problems go from rectangles to curved regions, switch the order to integrate functions with no elementary antiderivative, and end with the Gaussian integral and a joint density, the two places machine learning needs double integrals most.

- $\iint_R f\,dA$ is the integral of f over the region R; dA is the area of a small piece of R. When $f \geq 0$ it is the volume under the surface $z = f(x, y)$ above R, and $\iint_R 1\,dA$ is the area of R.
- **Iterated integrals.** If R is $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, then $\iint_R f\,dA = \displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx$. Do the inner integral first, treating x as a constant; its limits may depend on x. The outer limits are numbers.
- **Fubini.** On a rectangle $a \leq x \leq b$, $c \leq y \leq d$ the order does not matter, and if $f(x, y) = g(x)h(y)$ the integral is $\Big(\displaystyle\int_a^b g\,dx\Big)\Big(\int_c^d h\,dy\Big)$.
- **Switching the order.** Sketch the region and describe it the other way: $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$. The new limits come from the picture, not from the old limits.
- **Polar coordinates.** $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$, and $dA = r\,dr\,d\theta$: a small polar box has sides dr and $r\,d\theta$.
- **Joint densities.** If $f(x, y) \geq 0$ is the joint density of X and Y, then $\iint f\,dA = 1$ over the whole plane, $P\big((X, Y) \in R\big) = \iint_R f\,dA$, and $E[g(X, Y)] = \iint g\,f\,dA$. $\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y]$.

Builds on: Integration: substitution and parts, Parametric and polar curves: slopes, arc length and area

Problems

1. Evaluate $\displaystyle\int_0^2\int_0^1 (x+2y)\,dy\,dx$, then in the other order. difficulty ·

2. Evaluate $\displaystyle\iint_R xy^2\,dA$ over R: $0\le x\le 2$, $1\le y\le 3$. difficulty ·

3. Evaluate $\displaystyle\iint_T (x+y)\,dA$, where T is the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$. difficulty ··

4. Let R be the region between $y=x^2$ and $y=x$. Find its area and $\displaystyle\iint_R x\,dA$, in both orders. difficulty ··

5. Evaluate $\displaystyle\int_0^1\int_x^1 e^{y^2}\,dy\,dx$ by switching the order of integration. difficulty ··

6. Evaluate $\displaystyle\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\,dy\,dx$. difficulty ··

7. Evaluate $\displaystyle\iint_D (x^2+y^2)\,dA$ over the disk $x^2+y^2\le 4$. difficulty ··

8. Evaluate $\displaystyle\iint_Q xy\,dA$, where Q is the quarter disk $x\ge 0$, $y\ge 0$, $x^2+y^2\le 1$. difficulty ···

9. Show that $I=\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx=\sqrt{\pi}$ by computing I^2 as a double integral, and deduce the normalizing constant of the normal density with variance σ^2. difficulty ···

10. X and Y have joint density $f(x,y) = x + y$ on the unit square $0 \leq x, y \leq 1$, and 0 elsewhere. Check that it is a density, then find $P(X + Y \leq 1)$, $E[X]$, $E[XY]$ and $\mathrm{Cov}(X, Y)$. difficulty ···

Answers

1. 4
2. $\dfrac{52}{3}$
3. $\dfrac{1}{2}$
4. Area $\frac{1}{6}$; $\displaystyle\iint_R x\,dA = \frac{1}{12}$
5. $\dfrac{e-1}{2} \approx 0.859$
6. $\displaystyle\int_0^{\pi} \sin y\,dy = 2$
7. $8\pi \approx 25.13$
8. $\dfrac{1}{8}$
9. $\displaystyle\int_{-\infty}^{\infty} e^{-x^2/(2\sigma^2)}\,dx = \sigma\sqrt{2\pi}$, so the density is $\dfrac{1}{\sigma\sqrt{2\pi}} e^{-x^2/(2\sigma^2)}$
10. Density checks; $P(X+Y \le 1) = \frac{1}{3}$, $E[X] = \frac{7}{12}$, $E[XY] = \frac{1}{3}$, $\operatorname{Cov}(X,Y) = \frac{1}{3} - \frac{49}{144} = -\frac{1}{144}$

Worked solutions

Problem 1

> Evaluate $\int_0^2 \int_0^1 (x + 2y)\, dy\, dx$, then in the other order.

$\displaystyle\int_0^1 (x + 2y)\, dy = \Big[xy + y^2\Big]_0^1 = x + 1$	Inner integral in y, with x held constant.
$\displaystyle\int_0^2 (x + 1)\, dx = \Big[\tfrac{x^2}{2} + x\Big]_0^2 = 4$	Outer integral in x.
Other order: $\displaystyle\int_0^2 (x + 2y)\, dx = 2 + 4y$, and $\displaystyle\int_0^1 (2 + 4y)\, dy = 2 + 2 = 4$	On a rectangle the limits just trade places.
4	Both orders agree, as Fubini says. The rectangle has area 2, so the average of $x + 2y$ over it is 2, its value at the center $(1, \tfrac{1}{2})$.

Problem 2

> Evaluate $\iint_R xy^2\, dA$ over R: $0 \le x \le 2$, $1 \le y \le 3$.

$\displaystyle\iint_R xy^2\, dA = \Big(\int_0^2 x\, dx\Big)\Big(\int_1^3 y^2\, dy\Big)$	A product of a function of x and a function of y over a rectangle splits.
$\displaystyle\int_0^2 x\, dx = 2$ and $\displaystyle\int_1^3 y^2\, dy = \frac{27 - 1}{3} = \frac{26}{3}$	Two single integrals.
$\displaystyle\frac{52}{3}$	Check by iterating: the inner $\int_0^2 xy^2\, dx = 2y^2$, and $\int_1^3 2y^2\, dy = \frac{52}{3}$.

Problem 3

> Evaluate $\iint_T (x+y)\,dA$, where T is the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

T: $0 \le x \le 1$, $0 \le y \le x$	The triangle lies below the line $y = x$ and above the x-axis. For each x, y runs from 0 up to x.
$\displaystyle\int_0^x (x+y)\,dy = x^2 + \frac{x^2}{2} = \frac{3x^2}{2}$	The inner limit depends on x, so the result is a function of x.
$\displaystyle\int_0^1 \frac{3x^2}{2}\,dx = \frac{1}{2}$	Outer integral with number limits.
$\dfrac{1}{2}$	Check in the other order: T is also $0 \le y \le 1$, $y \le x \le 1$, and $\int_0^1\int_y^1 (x+y)\,dx\,dy = \int_0^1 \left(\frac{1}{2} + y - \frac{3}{2}y^2\right) dy = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = \frac{1}{2}$.

Problem 4

> Let R be the region between $y = x^2$ and $y = x$. Find its area and $\iint_R x\,dA$, in both orders.

The curves meet where $x^2 = x$, at $x = 0$ and $x = 1$, and $x \ge x^2$ between them	So R: $0 \le x \le 1$, $x^2 \le y \le x$.
Area $= \displaystyle\int_0^1 (x - x^2)\,dx = \frac{1}{6}$	The inner integral of 1 is the height of the strip.
$\displaystyle\int_0^1\int_{x^2}^x x\,dy\,dx = \int_0^1 (x^2 - x^3)\,dx = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$	x is a constant in the inner integral, so it multiplies the height $x - x^2$.
Other order: R is $0 \le y \le 1$, $y \le x \le \sqrt{y}$, and $\displaystyle\int_0^1\int_y^{\sqrt{y}} x\,dx\,dy = \int_0^1 \frac{y - y^2}{2}\,dy = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{12}$	Solve the curves for x: the line gives $x = y$ on the left and the parabola gives $x = \sqrt{y}$ on the right.

Area $\frac{1}{6}$; $\iint_R x\,dA = \frac{1}{12}$	The average of x over R is $\frac{1/12}{1/6} = \frac{1}{2}$: the centroid's x-coordinate.

Problem 5

> Evaluate $\int_0^1 \int_x^1 e^{y^2}\,dy\,dx$ by switching the order of integration.

e^{y^2} has no elementary antiderivative, so the inner integral cannot be done as written	Switching the order is the way out.
The region: $0 \le x \le 1$, $x \le y \le 1$, a triangle with vertices $(0,0)$, $(0,1)$ and $(1,1)$	For each x, y runs from the line $y = x$ up to 1.
Described the other way: $0 \le y \le 1$, $0 \le x \le y$	For each y, x runs from the y-axis to the line $x = y$.
$\int_0^1 \int_0^y e^{y^2}\,dx\,dy = \int_0^1 y\,e^{y^2}\,dy$	The inner integral of a constant in x is that constant times the length y, and the y is exactly what makes a substitution work.
$\frac{e-1}{2} \approx 0.859$	$u = y^2$, $du = 2y\,dy$: $\int_0^1 y e^{y^2}\,dy = \frac{1}{2}\left[e^u\right]_0^1$.

Problem 6

> Evaluate $\int_0^\pi \int_x^\pi \frac{\sin y}{y}\,dy\,dx$.

The region: $0 \le x \le \pi$, $x \le y \le \pi$, which is also $0 \le y \le \pi$, $0 \le x \le y$	The same triangle shape as Problem 5, scaled to π.

$\displaystyle\int_0^{\pi}\int_0^{y}\frac{\sin y}{y}\,dx\,dy=\int_0^{\pi}\frac{\sin y}{y}\cdot y\,dy$	$\frac{\sin y}{y}$ has no elementary antiderivative either; the inner integral in x supplies the factor y that cancels the denominator.
$\displaystyle\int_0^{\pi}\sin y\,dy=2$	$[-\cos y]_0^{\pi}=1+1$.

Problem 7

Evaluate $\displaystyle\iint_D (x^2+y^2)\,dA$ over the disk $x^2+y^2\le 4$.

D: $0\le r\le 2$, $0\le\theta\le 2\pi$, and $x^2+y^2=r^2$	A disk centered at the origin is a rectangle in r and θ.
$\displaystyle\iint_D (x^2+y^2)\,dA=\int_0^{2\pi}\int_0^{2} r^2\cdot r\,dr\,d\theta$	$dA=r\,dr\,d\theta$.
$\displaystyle\int_0^{2} r^3\,dr=4$, and $\displaystyle\int_0^{2\pi}4\,d\theta=8\pi$	The integrand does not depend on θ.
$8\pi\approx 25.13$	In x and y this would need $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}$, with square roots in the limits.

Problem 8

Evaluate $\displaystyle\iint_Q xy\,dA$, where Q is the quarter disk $x\ge 0$, $y\ge 0$, $x^2+y^2\le 1$.

Q: $0\le r\le 1$, $0\le\theta\le\frac{\pi}{2}$	The first quadrant is θ from 0 to $\frac{\pi}{2}$.
$xy=r^2\cos\theta\sin\theta$, so the integral is $\displaystyle\int_0^{\pi/2}\int_0^{1} r^3\cos\theta\sin\theta\,dr\,d\theta$	Including the r from dA.
$\displaystyle\left(\int_0^{1} r^3\,dr\right)\left(\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\right)=\frac{1}{4}\cdot\frac{1}{2}$	The integrand is a product and the limits are constants, so it splits; $\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta=\left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2}=\frac{1}{2}$.

$\dfrac{1}{8}$	Check in x and y: $\int_0^1 x \int_0^{\sqrt{1-x^2}} y \, dy \, dx = \int_0^1 \frac{x(1-x^2)}{2} \, dx = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{8}$.

Problem 9

Show that $I = \displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$ by computing I^2 as a double integral, and deduce the normalizing constant of the normal density with variance σ^2.

$I^2 = \displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA$	Give the second copy its own variable; a product of single integrals over the whole line is a double integral over the whole plane.
$\displaystyle\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r\, dr\, d\theta$	The whole plane is $0 \le r < \infty$, $0 \le \theta \le 2\pi$.
$\displaystyle\int_0^{\infty} r\, e^{-r^2}\,dr = \Big[-\tfrac{1}{2}e^{-r^2}\Big]_0^{\infty} = \frac{1}{2}$, so $I^2 = 2\pi \cdot \tfrac{1}{2} = \pi$	The r from dA is what makes this integral elementary.
$I = \sqrt{\pi}$	$I > 0$ because the integrand is positive.
$\displaystyle\int_{-\infty}^{\infty} e^{-x^2/(2\sigma^2)}\,dx = \sigma\sqrt{2\pi}$, **so the density is** $\dfrac{1}{\sigma\sqrt{2\pi}} e^{-x^2/(2\sigma^2)}$	Substitute $x = \sigma\sqrt{2}\,u$, $dx = \sigma\sqrt{2}\,du$: the integral is $\sigma\sqrt{2}\cdot\sqrt{\pi}$.

Problem 10

X and Y have joint density $f(x,y) = x + y$ on the unit square $0 \le x, y \le 1$, and 0 elsewhere. Check that it is a density, then find $P(X + Y \le 1)$, $E[X]$, $E[XY]$ and $\mathrm{Cov}(X, Y)$.

$\displaystyle\int_0^1\int_0^1 (x+y)\,dy\,dx = \int_0^1 \left(x + \frac{1}{2}\right) dx = 1$	It is nonnegative and integrates to 1.

$P(X+Y \le 1) = \int_0^1 \int_0^{1-x} (x+y)\,dy\,dx = \int_0^1 \left(x(1-x) + \frac{(1-x)^2}{2}\right)dx = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$	The region below the line $x+y=1$: for each x, y runs from 0 to $1-x$.
$E[X] = \int_0^1 \int_0^1 x(x+y)\,dy\,dx = \int_0^1 \left(x^2 + \frac{x}{2}\right)dx = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$	By symmetry $E[Y] = \frac{7}{12}$ too.
$E[XY] = \int_0^1 \int_0^1 xy(x+y)\,dy\,dx = \int_0^1 \left(\frac{x^2}{2} + \frac{x}{3}\right)dx = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$	$\iint x^2 y\,dA = \frac{1}{3}\cdot\frac{1}{2}$ and $\iint xy^2\,dA$ is the same.
Density checks; $P(X+Y \le 1) = \frac{1}{3}$, $E[X] = \frac{7}{12}$, $E[XY] = \frac{1}{3}$, $\mathrm{Cov}(X,Y) = \frac{1}{3} - \frac{49}{144} = -\frac{1}{144}$	Slightly negative: $E[Y \mid X=x] = \frac{x/2+1/3}{x+1/2}$ falls from $\frac{2}{3}$ at $x=0$ to $\frac{5}{9}$ at $x=1$, so a larger X comes with a slightly smaller Y on average. With $\mathrm{Var}\,X = \frac{5}{12} - \frac{49}{144} = \frac{11}{144}$, the correlation is $-\frac{1}{11}$.

Where this goes wrong

1. Switching the order by swapping the limits as written

The two pairs of limits are swapped along with dx and dy, as on a rectangle.

$\iint_R x\,dA$ over R: $0 \le x \le 1$, $x^2 \le y \le x$	Right so far: Problem 4, whose answer is $\frac{1}{12}$.
"For $dx\,dy$, x runs between the same two curves, with x renamed y."	The step that causes the mistake: the limits are rewritten without looking at the region.
$\int_0^1 \int_{y^2}^{y} x\,dx\,dy = \int_0^1 \frac{y^2 - y^4}{2}\,dy = \frac{1}{15}$ ✗	That describes the reflection of R across $y=x$, a different region. Solve each boundary for x: $y=x$ gives $x=y$ and $y=x^2$ gives $x=\sqrt{y}$, so the inner limits are y to $\sqrt{y}$, and the integral is $\frac{1}{12}$.

2. Integrating e^(y²) as if it had an antiderivative

The integral is attacked in the order it is written, with a guess for the antiderivative.

$\int_0^1 \int_x^1 e^{y^2}\, dy\, dx$	Right so far: Problem 5.
"The antiderivative of e^{y^2} is e^{y^2} divided by the derivative of y^2."	The habit that causes the mistake: the chain rule run backwards, which only works when that derivative is already a factor.
$\int e^{y^2}\, dy = \dfrac{e^{y^2}}{2y}$ ✗	Differentiate it: $\frac{d}{dy}\frac{e^{y^2}}{2y} = e^{y^2} - \frac{e^{y^2}}{2y^2}$, which at $y = 1$ is $\frac{e}{2} \approx 1.359$, not $e \approx 2.718$. No elementary antiderivative exists; switch the order, and the inner integral supplies the factor y, giving $\frac{e-1}{2}$.

3. Dropping the r in r dr dθ

$x^2 + y^2$ is replaced by r^2 and dA by $dr\, d\theta$, as if polar coordinates were a renaming.

$\iint_D (x^2 + y^2)\, dA$ over $x^2 + y^2 \le 4$, with $x^2 + y^2 = r^2$	Right so far: Problem 7.
"dA becomes $dr\, d\theta$."	The step that causes the mistake.
$\int_0^{2\pi} \int_0^2 r^2\, dr\, d\theta = \dfrac{16\pi}{3}$ ✗	A polar box of sides dr and $d\theta$ has area $r\, dr\, d\theta$, since the arc at radius r is $r\, d\theta$ long. Without it even the area comes out wrong: for the unit disk, $\int_0^{2\pi}\int_0^1 dr\, d\theta = 2\pi$, not π. With the r, the answer is 8π.

4. The full circle of θ for a quarter disk

$0 \le \theta \le 2\pi$ is the polar range used most often, and it is used again.

$\iint_Q xy\, dA$ with $xy = r^2 \cos\theta \sin\theta$ and $dA = r\, dr\, d\theta$	Right so far: Problem 8.

"θ goes from 0 to 2π."	The step that causes the mistake: the limits describe the whole disk, not the first quadrant.
$\int_0^{2\pi}\int_0^1 r^3\cos\theta\sin\theta\,dr\,d\theta = 0$ ✗	Over the whole disk, xy is negative in the second and fourth quadrants and cancels the first and third. A positive integrand on Q cannot give 0. The quarter disk is $0 \le \theta \le \frac{\pi}{2}$, and the integral is $\frac{1}{8}$.

5. Squaring the integral with one variable

$I^2 = I \cdot I$, and both copies are written with x.

$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$	Right so far: Problem 9.
"$I^2 = \int e^{-x^2} \cdot e^{-x^2}\,dx$."	The step that causes the mistake: the two integrals are merged into one.
$I^2 = \int_{-\infty}^{\infty} e^{-2x^2}\,dx = \sqrt{\pi/2} \approx 1.253$ ✗	A product of two integrals is not the integral of the product. Each copy needs its own dummy variable, $I^2 = \int e^{-x^2}dx \int e^{-y^2}dy$, and that is the double integral over the plane, equal to $\pi \approx 3.142$.

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