Differentiation rules: chain, product, quotient

Ten single-variable derivatives that exercise the product, quotient and chain rules on the functions machine learning uses — exponentials, logs, sigmoid, tanh, softplus, nested trig, x^x and an implicit curve — with worked solutions and the mistakes that drop the inner derivative or flip the quotient rule.

BEFORE YOU START

Three rules and a short table of basic derivatives are enough to differentiate every function on this page. The work goes wrong in a few places that recur: an inner derivative left off, a quotient rule with its numerator reversed, a product rule that multiplies the two derivatives, a power rule used on a variable exponent. These ten problems run from single rules to nested ones, then to x^x and a curve given by an equation.

- $f'(x)$ and $\frac{d}{dx}$ both mean the derivative with respect to x. u and v stand for any differentiable functions of x.
- Product rule: $(uv)' = u'v + uv'$.
- Quotient rule: $(u/v)' = (u'v - uv')/v^2$, where $v \neq 0$.
- Chain rule: $(f \circ g)' = f'(g)\,g'$.
- Basic derivatives: $(x^n)' = nx^{n-1}$ for a constant n; $(e^x)' = e^x$; $(\ln x)' = 1/x$ for $x > 0$; $(\sin x)' = \cos x$; $(\cos x)' = -\sin x$.
- $\ln$ is the natural log; σ is the logistic sigmoid, $\sigma(x) = 1/(1 + e^{-x})$; $\tanh x = (e^x - e^{-x})/(e^x + e^{-x})$. y names the function of x being differentiated when it is not given by an explicit formula (Problems 7 and 10).
- Write the derivative of the outer function *evaluated at the inner one*, then multiply by the inner derivative — the factor the mistakes below drop.

Problems

1. Differentiate $x^3 e^x$. difficulty ·

2. Differentiate $\dfrac{x}{1 + x^2}$. difficulty ·

3. Differentiate $\sin(3x^2)$. difficulty ·

4. Differentiate $\ln(1 + e^x)$ (softplus) and simplify. difficulty ··

5. Differentiate $\sigma(x) = \dfrac{1}{1 + e^{-x}}$ and write the result in terms of $\sigma(x)$ alone. difficulty ··

6. Differentiate $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ and write the result in terms of $\tanh x$. Then show $\tanh x = 2\sigma(2x) - 1$. difficulty ··

7. Differentiate x^x for $x > 0$. difficulty ··

8. Differentiate $\ln\left(\dfrac{e^x}{1 + e^x}\right)$ and simplify to a single sigmoid. difficulty ···

9. Differentiate $\sqrt{1 + \sin^2(2x)}$ and simplify with a double-angle identity. difficulty ···

10. The curve $x^2 + xy + y^2 = 7$ defines y as a function of x near most of its points. Find $\dfrac{dy}{dx}$. difficulty ···

Answers

1. $e^x(x^3 + 3x^2)$
2. $\dfrac{1 - x^2}{(1 + x^2)^2}$
3. $6x\cos(3x^2)$
4. $\sigma(x)$
5. $\sigma'(x) = \sigma(x)\big(1 - \sigma(x)\big)$
6. $1 - \tanh^2 x$; and $2\sigma(2x) - 1 = \dfrac{2}{1 + e^{-2x}} - 1 = \dfrac{1 - e^{-2x}}{1 + e^{-2x}} = \tanh x$
7. $x^x(\ln x + 1)$
8. $\sigma(-x)$
9. $\dfrac{\sin 4x}{\sqrt{1 + \sin^2(2x)}}$
10. $\dfrac{dy}{dx} = -\dfrac{2x + y}{x + 2y}$, valid where $x + 2y \neq 0$

Worked solutions

Problem 1

> Differentiate $x^3 e^x$.

$u = x^3$, $v = e^x$; $u' = 3x^2$, $v' = e^x$.	The function is a product of two factors whose derivatives are in the table, so the product rule applies.
$(x^3 e^x)' = 3x^2 \cdot e^x + x^3 \cdot e^x$.	Product rule, $u'v + uv'$: each term differentiates one factor and leaves the other alone.
$e^x(x^3 + 3x^2)$	Both terms contain e^x, so it factors out.

Problem 2

> Differentiate $\dfrac{x}{1+x^2}$.

$u = x$, $v = 1 + x^2$; $u' = 1$, $v' = 2x$.	The function is a quotient, so name the numerator and the denominator and differentiate each.
$\left(\dfrac{x}{1+x^2}\right)' = \dfrac{1 \cdot (1+x^2) - x \cdot 2x}{(1+x^2)^2}$.	Quotient rule: $u'v$ first, uv' subtracted, over v^2.
$1 \cdot (1+x^2) - x \cdot 2x = 1 + x^2 - 2x^2 = 1 - x^2$.	Expand the numerator and collect the x^2 terms.
$\dfrac{1 - x^2}{(1+x^2)^2}$	Sanity check: the function rises to its maximum $\frac{1}{2}$ at $x = 1$ and falls after it, and the derivative is 0 at $x = 1$, positive for $\lvert x \rvert < 1$ and negative for $x > 1$.

Problem 3

> Differentiate $\sin(3x^2)$.

Outer function sin, inner function $g(x) = 3x^2$.	sin is applied to $3x^2$, not to x, so this is a composition and needs the chain rule.
The derivative of sin, evaluated at the inner function: $\cos(3x^2)$.	$(\sin t)' = \cos t$, and the chain rule evaluates it at $t = 3x^2$.
The inner derivative: $g'(x) = 6x$.	Power rule with the constant 3 carried along.
$6x \cos(3x^2)$	Chain rule, $f'(g)\, g'$: the product of steps 2 and 3, with the factor $6x$ written first.

Problem 4

> Differentiate $\ln(1 + e^x)$ (softplus) and simplify.

Outer function ln, inner function $1 + e^x$.	The log is applied to $1 + e^x$, not to x, so the chain rule applies.
The derivative of ln, evaluated at the inner function: $\dfrac{1}{1 + e^x}$.	$(\ln t)' = 1/t$ at $t = 1 + e^x$, which is positive for every x.
The inner derivative: $(1 + e^x)' = e^x$.	The constant 1 has derivative 0.
$\dfrac{d}{dx} \ln(1 + e^x) = \dfrac{e^x}{1 + e^x}$.	Chain rule: step 2 times step 3.
$\dfrac{e^x}{1 + e^x} = \dfrac{1}{e^{-x} + 1}$.	Divide top and bottom by e^x, which is never 0, so that x appears only as e^{-x}, the form in the definition of σ.
$\sigma(x)$	$1/(1 + e^{-x})$ is the definition of $\sigma(x)$: softplus is an antiderivative of the sigmoid.

Problem 5

> Differentiate $\sigma(x) = \dfrac{1}{1+e^{-x}}$ and write the result in terms of $\sigma(x)$ alone.

$\sigma = (1+e^{-x})^{-1}.$	Written as a power of an inner function, σ needs only the chain rule and the power rule, not the quotient rule.
The derivative of t^{-1}, evaluated at the inner function: $-(1+e^{-x})^{-2}.$	$(t^{-1})' = -t^{-2}$, at $t = 1+e^{-x}$.
The inner derivative: $(1+e^{-x})' = e^{-x} \cdot (-1) = -e^{-x}.$	e^{-x} is itself a composition: the derivative of e^t at $t=-x$ is e^{-x}, and $(-x)' = -1$.
$\sigma'(x) = -(1+e^{-x})^{-2} \cdot (-e^{-x}) = \dfrac{e^{-x}}{(1+e^{-x})^2}.$	Chain rule: step 2 times step 3; the two minus signs cancel.
$\dfrac{e^{-x}}{(1+e^{-x})^2} = \sigma(x) \cdot \sigma(x)\, e^{-x}.$	Each factor $1/(1+e^{-x})$ is $\sigma(x)$, so the square in the denominator gives $\sigma(x)^2$.
$e^{-x} = \dfrac{1}{\sigma(x)} - 1.$	$1/\sigma(x) = 1+e^{-x}$ by the definition of σ; subtract 1.
$\sigma(x)^2\left(\dfrac{1}{\sigma(x)} - 1\right) = \sigma(x) - \sigma(x)^2.$	Substitute step 6 into step 5 and multiply out.
$\sigma'(x) = \sigma(x)\big(1-\sigma(x)\big)$	Factor out $\sigma(x)$. The derivative is largest, $\frac{1}{4}$, where $\sigma(x) = \frac{1}{2}$, at $x = 0$.

Problem 6

> Differentiate $\tanh x = \dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$ and write the result in terms of $\tanh x$. Then show $\tanh x = 2\sigma(2x) - 1$.

$u = e^{x}-e^{-x}$, $v = e^{x}+e^{-x}$; $u' = e^{x}+e^{-x}$, $v' = e^{x}-e^{-x}$.	$(e^{-x})' = -e^{-x}$ (Problem 5, step 3), so differentiating flips the sign of each e^{-x} term: $u' = v$ and $v' = u$.

$\dfrac{d}{dx}\tanh x = \dfrac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}.$	Quotient rule: $u'v = v^2$ and $uv' = u^2$.
$\dfrac{v^2 - u^2}{v^2} = 1 - \left(\dfrac{u}{v}\right)^2.$	Split the fraction into two terms; the first is 1.
$\dfrac{d}{dx}\tanh x = 1 - \tanh^2 x.$	u/v is $\tanh x$ by definition.
$2\sigma(2x) - 1 = \dfrac{2}{1 + e^{-2x}} - 1 = \dfrac{2 - (1 + e^{-2x})}{1 + e^{-2x}} = \dfrac{1 - e^{-2x}}{1 + e^{-2x}}.$	The definition of σ at $2x$, then put 1 over the same denominator.
$\dfrac{1 - e^{-2x}}{1 + e^{-2x}} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}.$	Multiply top and bottom by e^x: $e^x \cdot e^{-2x} = e^{-x}$.
$1 - \tanh^2 x$; **and** $2\sigma(2x) - 1 = \dfrac{2}{1 + e^{-2x}} - 1 = \dfrac{1 - e^{-2x}}{1 + e^{-2x}} = \tanh x$	Steps 4 to 6. So $\tanh$ is a sigmoid stretched to the range $(-1, 1)$ and squeezed horizontally by 2.

Problem 7

> Differentiate x^x for $x > 0$.

Let $y = x^x$. Then $\ln y = x \ln x$.	The exponent is the variable, so the power rule, which needs a constant exponent, does not apply; the log turns the exponent into a factor. $x > 0$ makes $y > 0$, so $\ln y$ is defined.
$\dfrac{d}{dx}\ln y = \dfrac{1}{y} \cdot y'.$	y is a function of x, so this is a chain rule: the derivative of $\ln$ evaluated at y is $1/y$, times the inner derivative y'.
$\dfrac{d}{dx}(x \ln x) = 1 \cdot \ln x + x \cdot \dfrac{1}{x} = \ln x + 1.$	Product rule with $u = x$ and $v = \ln x$; $(\ln x)' = 1/x$ for $x > 0$.
$\dfrac{y'}{y} = \ln x + 1$, so $y' = y(\ln x + 1).$	The two sides of step 1 are equal for every $x > 0$, so their derivatives are equal; multiply by y.

$x^x(\ln x + 1)$	Substitute $y = x^x$.

Another route: through the exponential

$x^x = e^{x \ln x}$ for $x > 0$.	$x = e^{\ln x}$, and raising both sides to the power x multiplies the exponent by x.
The derivative of e^t, evaluated at the inner function: $e^{x \ln x}$.	$(e^t)' = e^t$ at $t = x \ln x$.
The inner derivative: $(x \ln x)' = \ln x + 1$.	Product rule, as in step 3 of the first route.
$x^x(\ln x + 1)$	Chain rule: step 2 times step 3, and $e^{x \ln x} = x^x$.

Problem 8

> Differentiate $\ln\left(\dfrac{e^x}{1+e^x}\right)$ and simplify to a single sigmoid.

$\ln \dfrac{e^x}{1+e^x} = \ln e^x - \ln(1+e^x) = x - \ln(1+e^x)$.	Simplify before differentiating: the log of a quotient is a difference of logs, and $\ln e^x = x$. That replaces a chain rule wrapped around a quotient rule with two terms whose derivatives are already known.
$\dfrac{d}{dx}\big(x - \ln(1+e^x)\big) = 1 - \sigma(x)$.	x has derivative 1, and the derivative of $\ln(1+e^x)$ is $\sigma(x)$ (Problem 4).
$1 - \sigma(x) = 1 - \dfrac{1}{1+e^{-x}} = \dfrac{e^{-x}}{1+e^{-x}}$.	Put 1 over the denominator $1+e^{-x}$.
$\dfrac{e^{-x}}{1+e^{-x}} = \dfrac{1}{e^x+1}$.	Divide top and bottom by e^{-x}.

$\sigma(-x)$	$1/(1+e^x)$ is the definition of σ evaluated at $-x$. Since $e^x/(1+e^x) = \sigma(x)$ (Problem 4, step 5), this also says $\frac{d}{dx}\ln\sigma(x) = 1-\sigma(x) = \sigma(-x)$.

Problem 9

> Differentiate $\sqrt{1+\sin^2(2x)}$ and simplify with a double-angle identity.

Four layers, outermost first: $\sqrt{t}$, then $1+s^2$, then $\sin w$, then $w = 2x$.	Reading the function from the outside in lists the compositions; the chain rule needs one factor per layer.
The derivative of $\sqrt{t}$, evaluated at the inner function $1+\sin^2 2x$: $\dfrac{1}{2\sqrt{1+\sin^2 2x}}$.	$(\sqrt{t})' = 1/(2\sqrt{t})$ at $t = 1+\sin^2 2x$, which is at least 1, so the square root is never 0.
The derivative of $1+s^2$, evaluated at $\sin 2x$: $2\sin 2x$.	$(1+s^2)' = 2s$ at $s = \sin 2x$.
The derivative of $\sin w$, evaluated at $2x$: $\cos 2x$; and $(2x)' = 2$.	$(\sin w)' = \cos w$ at $w = 2x$, and the innermost derivative is the constant 2.
$\dfrac{d}{dx}\sqrt{1+\sin^2 2x} = \dfrac{2\sin 2x\cos 2x\cdot 2}{2\sqrt{1+\sin^2 2x}}$.	Chain rule through all four layers: the product of the factors in steps 2 to 4.
$2\sin 2x\cos 2x = \sin 4x$.	Double-angle identity $\sin 2\theta = 2\sin\theta\cos\theta$ with $\theta = 2x$.
$\dfrac{\sin 4x}{\sqrt{1+\sin^2(2x)}}$	Substitute step 6 into the numerator of step 5; the remaining factor 2 cancels the 2 in the denominator.

Problem 10

> The curve $x^2 + xy + y^2 = 7$ defines y as a function of x near most of its points. Find $\dfrac{dy}{dx}$.

Treat y as a function of x and differentiate both sides of $x^2 + xy + y^2 = 7$ with respect to x.	The equation holds at every x in a stretch of the curve, so the derivatives of the two sides are equal there. Write y' for dy/dx.
$\dfrac{d}{dx} x^2 = 2x.$	Power rule.
$\dfrac{d}{dx}(xy) = 1 \cdot y + x\,y'.$	Product rule with $u = x$ and $v = y$; y depends on x, so its derivative y' stays.
$\dfrac{d}{dx} y^2 = 2y \cdot y'.$	Chain rule: the derivative of t^2 evaluated at y is $2y$, times the inner derivative y'.
$2x + y + xy' + 2yy' = 0.$	Add steps 2 to 4; the right side, 7, is constant.
$y'(x + 2y) = -(2x + y).$	Collect the terms containing y' on one side.
$\dfrac{dy}{dx} = -\dfrac{2x + y}{x + 2y}$, **valid where** $x + 2y \neq 0$	Divide by $x + 2y$, which is allowed only where it is not 0. Where $x + 2y = 0$ the curve meets $y = -x/2$, so $\frac{3}{4}x^2 = 7$ and $x = \pm\sqrt{28/3}$; the tangent there is vertical, and near those two points y is not a function of x.

Where this goes wrong

1. Forgetting the inner derivative

$\frac{d}{dx}\sin x = \cos x$ is the first derivative most people memorize, and it is easy to reuse it with anything in the place of x.

$\sin(3x^2)$ has outer function sin and inner function $3x^2$	Right so far: the function is a composition.

“The derivative of sin is cos, so the derivative of sin(anything) is cos(anything).”	The analogy that causes the mistake: $\frac{d}{dx}\sin x = \cos x$ applied with $3x^2$ in the place of x.
$\frac{d}{dx}\sin(3x^2) = \cos(3x^2)$ ✗	This is the outer derivative evaluated at the inner function, the chain rule's first factor, with the second factor, $(3x^2)' = 6x$, left off. The correct derivative is $6x\cos(3x^2)$ (Problem 3). $\frac{d}{dx}\sin x = \cos x$ is the case where the inner function is x and its derivative is 1, which is why the factor never shows up there.

2. Quotient rule with the numerator the wrong way round

The product rule's two terms can be added in either order, and the quotient rule looks like it has the same two terms.

$u = x$, $v = 1 + x^2$, $u' = 1$, $v' = 2x$	Right so far: the pieces are those of Problem 2.
“The numerator is the two cross terms, one minus the other.”	The analogy that causes the mistake: remembering the terms of the product rule without the order the minus sign imposes.
$\left(\frac{x}{1+x^2}\right)' = \frac{x \cdot 2x - (1+x^2)}{(1+x^2)^2}$ ✗	That is $uv' - u'v$; the quotient rule is $u'v - uv'$. Swapping the two terms of a difference flips its sign, so the whole answer comes out negated: $\frac{x^2-1}{(1+x^2)^2}$ instead of $\frac{1-x^2}{(1+x^2)^2}$. A check at $x = 0$ catches it: the function rises through the origin, so the derivative there must be positive, and this line gives -1.

3. Product rule as the product of the derivatives

For a sum, the derivative is the sum of the derivatives, and it is natural to expect the same for a product.

$u = x^3$, $v = e^x$, $u' = 3x^2$, $v' = e^x$	Right so far: the pieces are those of Problem 1.
"The derivative of a product is the product of the derivatives."	The analogy that causes the mistake: the sum rule carried over to products.
$(x^3 e^x)' = 3x^2 e^x$ ✗	That is $u'v'$, and because $v' = e^x = v$ it also equals $u'v$, one of the product rule's two terms, which makes it look half-right. The other term is $uv' = x^3 e^x$, and the derivative is $e^x(x^3 + 3x^2)$ (Problem 1). A check with $x \cdot x = x^2$ shows the analogy fails: the product of the derivatives is $1 \cdot 1 = 1$, but $(x^2)' = 2x$.

4. Power rule on a variable exponent

x^x looks like x^n, and $(x^n)' = nx^{n-1}$ is the rule people reach for first.

x^x for $x > 0$	Right so far: the function and its domain.
"x^x is x to a power, so bring the power down and subtract 1."	The analogy that causes the mistake: the power rule applied with $n = x$.
$(x^x)' = x \cdot x^{x-1}$ ✗	The power rule needs a constant exponent; here the exponent changes with x, and its contribution, the term $x^x \ln x$, is lost. This line simplifies to x^x, which is not the derivative: at $x = 1$ it gives 1, the correct value, but at $x = e$ it gives e^e where the derivative is $2e^e$. Take logs first (Problem 7): $x^x(\ln x + 1)$.

5. Derivative of a log without the inside

$(\ln x)' = 1/x$ is short and easy to reuse with any argument in the place of x.

$\ln(1 + e^x)$ has outer function $\ln$ and inner function $u = 1 + e^x$	Right so far: the function is a composition.
"The derivative of $\ln$ of something is one over that something."	The analogy that causes the mistake: $(\ln x)' = 1/x$ applied with $1 + e^x$ in the place of x.
$\frac{d}{dx}\ln(1 + e^x) = \frac{1}{1 + e^x}$ ✗	This is only the outer derivative evaluated at the inner function. The chain rule gives $(\ln u)' = u'/u$, and $u' = e^x$ here, so the derivative is $\frac{e^x}{1 + e^x} = \sigma(x)$ (Problem 4). The wrong line is $\sigma(-x)$, which falls where the true derivative rises: softplus is increasing, and its slope must approach 1 as x grows, not 0.

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