Complex numbers in polar form: Euler's formula, De Moivre and roots of unity

Ten problems on complex numbers in polar form: the modulus and argument of −1 + √3i and −2 − 2i, polar back to rectangular, a product and a quotient that give cos(5π/12) exactly, (1 + i)^10 and (1 − √3i)^6 by De Moivre, the three cube roots of 8i, the fifth roots of unity and cos(2π/5), the triple-angle formulas for cos 3θ and sin 3θ, the integral of e^x cos x from the real part of e^((1+i)x), a rotation about a point other than the origin, and the rotary position embeddings that make an attention score depend only on the distance between positions, with worked solutions and the mistakes of taking arctan(b/a) in the wrong quadrant, adding the moduli of a product, keeping only one of the n roots, raising the real and imaginary parts to a power separately and taking the modulus of a sum as the sum of the moduli.

BEFORE YOU START

A complex number $a + bi$ is a point in the plane, and like any point it can be given by its distance from the origin and its angle instead of by its coordinates. In that polar form, $z = re^{i\theta}$, the hard operations become easy: multiplying multiplies the distances and adds the angles, a power is a power of the distance and a multiple of the angle, and the n-th roots of a number sit evenly around a circle. Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ is what connects the two forms, and it turns trigonometry into algebra: triple-angle formulas, sums of cosines and integrals of $e^x\cos x$ all come out of the rules for exponents. The same rotations are inside the discrete Fourier transform, the solutions of oscillating differential equations and the rotary position embeddings of transformer models. These ten problems convert between the forms, multiply, take powers and roots, and use the result.

- **Polar form.** $z = a + bi = r(\cos\theta + i\sin\theta) = re^{i\theta}$, with modulus $r = |z| = \sqrt{a^2 + b^2}$ and argument θ. Angles on this page are in **radians**.
- **The argument needs the quadrant.** $\tan\theta = b/a$ does not decide θ: $\arctan(b/a)$ lies in $(-\pi/2, \pi/2)$, so when $a < 0$ add or subtract π. Always check that $\cos\theta$ has the sign of a and $\sin\theta$ the sign of b.
- **Euler's formula.** $e^{i\theta} = \cos\theta + i\sin\theta$, so $|e^{i\theta}| = 1$, $e^{i\pi} = -1$ and $e^{i(\theta+2\pi)} = e^{i\theta}$. Conversely $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$.
- **Products and quotients.** $r_1e^{i\theta_1}\cdot r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$ and $\dfrac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \dfrac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$: moduli multiply, arguments add.
- **De Moivre.** $(re^{i\theta})^n = r^ne^{in\theta}$, that is $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.
- **Roots.** The equation $z^n = re^{i\theta}$ with $r > 0$ has exactly n solutions, $z_k = r^{1/n}e^{i(\theta+2\pi k)/n}$ for $k = 0, 1, \ldots, n-1$, spaced $2\pi/n$ apart on a circle. The n-th roots of unity $e^{2\pi ik/n}$ add up to 0 for $n \geq 2$.

- **Rotation.** Multiplying by $e^{i\alpha}$ rotates a point by α counterclockwise about the origin; $\bar{z}$ reflects it in the real axis, and $z\bar{z} = |z|^2$.

Builds on: Trig identities, Solving trig equations and the unit circle

Problems

1. Write $z = -1 + \sqrt{3}\,i$ and $w = -2 - 2i$ in polar form $re^{i\theta}$, with $-\pi < \theta \le \pi$. difficulty ·

2. Write $4e^{5\pi i/6}$, $3e^{-i\pi/2}$ and $e^{i\pi} + 1$ in the form $a + bi$. difficulty ·

3. Let $z = 1 + i$ and $w = \sqrt{3} + i$. Find zw and z/w in polar form, then multiply zw out in rectangular form and use it to find $\cos\frac{5\pi}{12}$ exactly. difficulty ··

4. Compute $(1+i)^{10}$ and $(1 - \sqrt{3}\,i)^{6}$ with De Moivre's theorem. difficulty ··

5. Find all solutions of $z^3 = 8i$ in polar and rectangular form, and check that they add up to 0. difficulty ··

6. Let $\omega = e^{2\pi i/5}$. Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$, and use its real part to find $\cos\frac{2\pi}{5}$ exactly. difficulty ··

7. Expand $(\cos\theta + i\sin\theta)^3$ and use De Moivre's theorem to write $\cos 3\theta$ in terms of $\cos\theta$ and $\sin 3\theta$ in terms of $\sin\theta$. Check both at $\theta = \pi/9$. difficulty ··

8. Compute $\int_0^{\pi} e^{x}\cos x\,dx$ and $\int_0^{\pi} e^{x}\sin x\,dx$ at once, as the real and imaginary parts of $\int_0^{\pi} e^{(1+i)x}\,dx$. difficulty ···

9. Rotate the point $P = (4, 2)$ by 60° counterclockwise about the point $C = (1, 1)$. Check that its distance from C does not change. difficulty ··

10. Rotary position embeddings (RoPE) treat each pair of coordinates of a query or key as a complex number and rotate the vector at position m by the angle $m\theta$. Take the pair $q = (1, 2)$, the pair $k = (3, -1)$ and $\theta = \pi/6$. Show that the score $\operatorname{Re}\left(q_m \overline{k_n}\right)$ between the rotated query $q_m = q e^{im\theta}$ and the rotated key $k_n = k e^{in\theta}$ is the ordinary dot product when $m = n$, and find it for $(m, n) = (5, 2)$ and $(8, 5)$. difficulty ···

Answers

1. $z = 2e^{2\pi i/3}$ and $w = 2\sqrt{2}\,e^{-3\pi i/4}$
2. $-2\sqrt{3} + 2i \approx -3.464 + 2i$, $-3i$, and $e^{i\pi} + 1 = 0$
3. $zw = 2\sqrt{2}\,e^{5\pi i/12}$, $z/w = \frac{\sqrt{2}}{2}e^{i\pi/12}$, and $\cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4} \approx 0.2588$
4. $(1+i)^{10} = 32i$ and $(1-\sqrt{3}\,i)^{6} = 64$
5. $z = \sqrt{3}+i$, $-\sqrt{3}+i$, $-2i$; their sum is 0
6. $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4} \approx 0.3090$
7. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$
8. $\int_0^{\pi} e^x \cos x\,dx = -\frac{e^{\pi}+1}{2} \approx -12.07$ and $\int_0^{\pi} e^x \sin x\,dx = \frac{e^{\pi}+1}{2} \approx 12.07$
9. $P' = \left(\frac{5-\sqrt{3}}{2},\ \frac{3+3\sqrt{3}}{2}\right) \approx (1.634, 4.098)$
10. The score is $\operatorname{Re}\big((1+7i)i\big) = \operatorname{Re}(-7+i) = -7$ for both $(5,2)$ and $(8,5)$

Worked solutions

Problem 1

> Write $z = -1 + \sqrt{3}\,i$ and $w = -2 - 2i$ in polar form $re^{i\theta}$, with $-\pi < \theta \le \pi$.

$\lvert z\rvert = \sqrt{1+3} = 2$	The distance from the origin to $(-1, \sqrt{3})$.
$\cos\theta = -\tfrac{1}{2}$, $\sin\theta = \tfrac{\sqrt{3}}{2}$	Divide the coordinates by r. Cosine negative and sine positive: the second quadrant.
$\theta = \tfrac{2\pi}{3}$	The reference angle is $\pi/3$; in the second quadrant that is $\pi - \pi/3$.
$\lvert w\rvert = \sqrt{4+4} = 2\sqrt{2}$, and $\theta = -\tfrac{3\pi}{4}$	Both coordinates negative: the third quadrant, reference angle $\pi/4$, measured clockwise from the positive real axis.
$z = 2e^{2\pi i/3}$ **and** $w = 2\sqrt{2}\,e^{-3\pi i/4}$	A calculator's $\arctan(\sqrt{3}/(-1)) = -\pi/3$ points the opposite way, into the fourth quadrant (Mistake 1). The angle $5\pi/4$ for w is also right; it is the same direction outside the range asked for.

Problem 2

> Write $4e^{5\pi i/6}$, $3e^{-i\pi/2}$ and $e^{i\pi} + 1$ in the form $a + bi$.

$4e^{5\pi i/6} = 4\big(\cos\tfrac{5\pi}{6} + i\sin\tfrac{5\pi}{6}\big)$	Euler's formula.
$= 4\big(-\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i\big) = -2\sqrt{3} + 2i$	$5\pi/6$ is in the second quadrant, reference angle $\pi/6$.
$3e^{-i\pi/2} = 3(0 - i) = -3i$	A quarter turn clockwise from the positive real axis lands on the negative imaginary axis.
$e^{i\pi} = \cos\pi + i\sin\pi = -1$	Half a turn.

$-2\sqrt{3}+2i \approx -3.464+2i$, $-3i$, **and** $e^{i\pi}+1=0$	Check the first: $(2\sqrt{3})^2+2^2=16=4^2$.

Problem 3

> Let $z=1+i$ and $w=\sqrt{3}+i$. Find zw and z/w in polar form, then multiply zw out in rectangular form and use it to find $\cos\frac{5\pi}{12}$ exactly.

$z=\sqrt{2}\,e^{i\pi/4}$, $w=2e^{i\pi/6}$	Both are in the first quadrant: $\arctan 1=\pi/4$ and $\arctan(1/\sqrt{3})=\pi/6$.
$zw=2\sqrt{2}\,e^{5\pi i/12}$	Multiply the moduli, $\sqrt{2}\cdot 2$, and add the arguments, $\pi/4+\pi/6=5\pi/12$.
$z/w=\frac{\sqrt{2}}{2}e^{i\pi/12}$	Divide the moduli and subtract the arguments: $\pi/4-\pi/6=\pi/12$.
$zw=(\sqrt{3}-1)+(\sqrt{3}+1)i$	Directly: $\sqrt{3}+i+\sqrt{3}\,i+i^2$.
$zw=2\sqrt{2}\,e^{5\pi i/12}$, $z/w=\frac{\sqrt{2}}{2}e^{i\pi/12}$, **and** $\cos\frac{5\pi}{12}=\frac{\sqrt{3}-1}{2\sqrt{2}}=\frac{\sqrt{6}-\sqrt{2}}{4}\approx 0.2588$	The real part of zw is $2\sqrt{2}\cos\frac{5\pi}{12}$; divide by $2\sqrt{2}$. Adding the moduli instead gives $\sqrt{2}+2\approx 3.414$, not $2\sqrt{2}\approx 2.828$ (Mistake 2).

Problem 4

> Compute $(1+i)^{10}$ and $(1-\sqrt{3}\,i)^6$ with De Moivre's theorem.

$1+i=\sqrt{2}\,e^{i\pi/4}$	Modulus $\sqrt{2}$, first quadrant.
$(1+i)^{10}=(\sqrt{2})^{10}e^{10\pi i/4}=32e^{5\pi i/2}$	Raise the modulus to the power, multiply the angle by it.
$e^{5\pi i/2}=e^{i\pi/2}=i$	Subtract a full turn, 2π.
$1-\sqrt{3}\,i=2e^{-i\pi/3}$, so $(1-\sqrt{3}\,i)^6=64e^{-2\pi i}$	Fourth quadrant, reference angle $\pi/3$; $2^6=64$.

$(1+i)^{10} = 32i$ **and** $(1-\sqrt{3}\,i)^{6} = 64$	$e^{-2\pi i} = 1$. Check the first by squaring: $(1+i)^2 = 2i$, and $(2i)^5 = 32i^5 = 32i$. Taking $1^{10} + i^{10} = 0$ instead is Mistake 4.

Problem 5

> Find all solutions of $z^3 = 8i$ in polar and rectangular form, and check that they add up to 0.

$8i = 8e^{i\pi/2}$	On the positive imaginary axis.
$z_k = 2e^{i(\pi/2 + 2\pi k)/3}$ for $k = 0, 1, 2$	The cube root of the modulus, and the argument divided by 3 after adding $2\pi k$.
$z_0 = 2e^{i\pi/6} = \sqrt{3} + i$	$\pi/6$: $2\cos\frac{\pi}{6} = \sqrt{3}$ and $2\sin\frac{\pi}{6} = 1$.
$z_1 = 2e^{5\pi i/6} = -\sqrt{3} + i$ and $z_2 = 2e^{3\pi i/2} = -2i$	Each root is $2\pi/3$ further around the circle of radius 2.
$z = \sqrt{3} + i,\ -\sqrt{3} + i,\ -2i$; **their sum is** 0	The real parts cancel and $1 + 1 - 2 = 0$. Check one: $(-2i)^3 = -8i^3 = 8i$. Stopping at z_0 loses two of the three roots (Mistake 3).

Problem 6

> Let $\omega = e^{2\pi i/5}$. Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$, and use its real part to find $\cos\frac{2\pi}{5}$ exactly.

$(\omega - 1)(1 + \omega + \cdots + \omega^4) = \omega^5 - 1 = 0$	The telescoping product, and $\omega^5 = e^{2\pi i} = 1$.
$1 + \omega + \cdots + \omega^4 = 0$	$\omega \neq 1$, so the other factor is 0.
$1 + 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = 0$	Real parts: $\omega^4 = \bar{\omega}$ and $\omega^3 = \overline{\omega^2}$ pair up, and each pair contributes twice a cosine.
With $c = \cos\frac{2\pi}{5}$: $1 + 2c + 2(2c^2 - 1) = 0$, so $4c^2 + 2c - 1 = 0$	The double-angle formula $\cos 2x = 2\cos^2 x - 1$.

$\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4} \approx 0.3090$	The positive root of the quadratic, since $2\pi/5 < \pi/2$. The five roots each have modulus 1, yet their sum has modulus 0, not 5 (Mistake 5).

Problem 7

> Expand $(\cos\theta + i\sin\theta)^3$ and use De Moivre's theorem to write $\cos 3\theta$ in terms of $\cos\theta$ and $\sin 3\theta$ in terms of $\sin\theta$. Check both at $\theta = \pi/9$.

$(c + is)^3 = c^3 + 3ic^2 s - 3cs^2 - is^3$	The binomial expansion with $c = \cos\theta$, $s = \sin\theta$; $i^2 = -1$ and $i^3 = -i$.
$\cos 3\theta = c^3 - 3cs^2$	De Moivre: the left side is $\cos 3\theta + i\sin 3\theta$. Match real parts.
$\cos 3\theta = 4c^3 - 3c$	Replace s^2 by $1 - c^2$.
$\sin 3\theta = 3c^2 s - s^3 = 3s - 4s^3$	Match imaginary parts and replace c^2 by $1 - s^2$.
$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ **and** $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$	At $\theta = \pi/9$, $\cos\theta \approx 0.9397$ gives $4(0.8298) - 2.8191 \approx 0.5 = \cos\frac{\pi}{3}$, and $\sin\theta \approx 0.3420$ gives $\approx 0.8660 = \sin\frac{\pi}{3}$.

Problem 8

> Compute $\int_0^\pi e^x \cos x\,dx$ and $\int_0^\pi e^x \sin x\,dx$ at once, as the real and imaginary parts of $\int_0^\pi e^{(1+i)x}\,dx$.

$e^{(1+i)x} = e^x\cos x + ie^x \sin x$	Euler's formula, times the real factor e^x.
$\int_0^\pi e^{(1+i)x}\,dx = \dfrac{e^{(1+i)\pi} - 1}{1+i}$	The exponential rule works with a complex constant: the derivative of e^{kx} is ke^{kx}.
$e^{(1+i)\pi} = e^\pi e^{i\pi} = -e^\pi$	So the numerator is $-(e^\pi + 1)$.
$\dfrac{1}{1+i} = \dfrac{1-i}{2}$	Multiply top and bottom by the conjugate: $(1+i)(1-i) = 2$.

$\int_0^{\pi} e^x \cos x \, dx = -\frac{e^{\pi}+1}{2} \approx -12.07$ **and** $\int_0^{\pi} e^x \sin x \, dx = \frac{e^{\pi}+1}{2} \approx 12.07$	$-(e^{\pi}+1)\frac{1-i}{2}$ has real part $-\frac{e^{\pi}+1}{2}$ and imaginary part $+\frac{e^{\pi}+1}{2}$. Integrating by parts twice gives the same, with more work.

Problem 9

> Rotate the point $P = (4, 2)$ by 60° counterclockwise about the point $C = (1, 1)$. Check that its distance from C does not change.

$P - C = 3 + i$	Move C to the origin first: rotation by $e^{i\alpha}$ only turns about 0.
$e^{i\pi/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$	60° is $\pi/3$ radians.
$(3+i)e^{i\pi/3} = \left(\frac{3}{2} - \frac{\sqrt{3}}{2}\right) + \left(\frac{3\sqrt{3}}{2} + \frac{1}{2}\right)i$	Multiply out: $\frac{3}{2} + \frac{3\sqrt{3}}{2}i + \frac{1}{2}i - \frac{\sqrt{3}}{2}$.
Add $C = 1 + i$ back	Undo the shift.
$P' = \left(\frac{5-\sqrt{3}}{2},\ \frac{3+3\sqrt{3}}{2}\right) \approx (1.634, 4.098)$	Its distance from C is $\lvert 3 + i \rvert = \sqrt{10}$ before and after, since $\lvert e^{i\pi/3} \rvert = 1$: $0.634^2 + 3.098^2 \approx 10$. Rotating $4 + 2i$ itself would turn it about the origin instead.

Problem 10

> Rotary position embeddings (RoPE) treat each pair of coordinates of a query or key as a complex number and rotate the vector at position m by the angle $m\theta$. Take the pair $q = (1, 2)$, the pair $k = (3, -1)$ and $\theta = \pi/6$. Show that the score $\operatorname{Re}\left(q_m \overline{k_n}\right)$ between the rotated query $q_m = q e^{im\theta}$ and the rotated key $k_n = k e^{in\theta}$ is the ordinary dot product when $m = n$, and find it for $(m, n) = (5, 2)$ and $(8, 5)$.

$\operatorname{Re}(z\bar{w}) = ac + bd$ for $z = a + bi$, $w = c + di$	$(a + bi)(c - di)$ has real part $ac + bd$: the dot product of the two pairs.
$q_m \overline{k_n} = q\bar{k}\, e^{i(m-n)\theta}$	$\overline{e^{in\theta}} = e^{-in\theta}$, so the two rotations combine into one by the difference of the angles.

$q\bar{k} = (1+2i)(3+i) = 1+7i$	$3 + i + 6i + 2i^2$. With $m = n$ the score is $\mathrm{Re}(1+7i) = 1 = 1 \cdot 3 + 2 \cdot (-1)$.
$(m-n)\theta = 3 \cdot \frac{\pi}{6} = \frac{\pi}{2}$ for both pairs	$5 - 2 = 8 - 5 = 3$.
The score is $\mathrm{Re}\left((1+7i)i\right) = \mathrm{Re}(-7+i) = -7$ **for both** $(5,2)$ **and** $(8,5)$	It depends only on the offset $m-n$, not on where the pair sits in the sequence, which is why RoPE encodes relative position. The rotations keep $\lvert q\rvert = \sqrt{5}$ and $\lvert k\rvert = \sqrt{10}$, so the norms are unchanged.

Where this goes wrong

1. arctan(b/a) in the wrong quadrant

The argument satisfies $\tan\theta = b/a$, so the inverse tangent looks like it gives the angle directly.

$z = -1 + \sqrt{3}\,i$, $\lvert z\rvert = 2$	Right so far: Problem 1.
"$\theta = \arctan\dfrac{\sqrt{3}}{-1}$."	arctan only returns angles in $(-\pi/2, \pi/2)$, the right half of the plane, but z has a negative real part.
$z = 2e^{-i\pi/3}$ ✗	$2e^{-i\pi/3} = 1 - \sqrt{3}\,i$, which is $-z$, the point directly opposite. Adding π gives the right angle, $-\pi/3 + \pi = 2\pi/3$, and $2e^{2\pi i/3} = -1 + \sqrt{3}\,i$.

2. Adding the moduli of a product

Arguments add when complex numbers are multiplied, and it is easy to add the moduli along with them.

$z = \sqrt{2}\,e^{i\pi/4}$, $w = 2e^{i\pi/6}$, and the argument of zw is $5\pi/12$	Right so far: Problem 3.

“The moduli add too.”	Only the exponents add: $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$. The moduli are ordinary positive factors in front, and they multiply.
$zw = (\sqrt{2}+2)e^{5\pi i/12}$, modulus ≈ 3.414 ✗	$zw = (\sqrt{3}-1)+(\sqrt{3}+1)i$ has modulus $\sqrt{(\sqrt{3}-1)^2+(\sqrt{3}+1)^2} = \sqrt{8} = 2\sqrt{2} \approx 2.828$.

3. Only one of the n roots

A calculator or a single application of the root formula returns one cube root, and it looks like the answer.

$8i = 8e^{i\pi/2}$	Right so far: Problem 5.
“$z = 8^{1/3}e^{i\pi/6}$, and that is the solution.”	The argument of $8i$ is also $\pi/2 + 2\pi$ and $\pi/2 + 4\pi$; divided by 3 these give different directions.
$z^3 = 8i$ has the one solution $\sqrt{3}+i$ ✗	A cubic has three roots: $\sqrt{3}+i$, $-\sqrt{3}+i$ and $-2i$, spaced $2\pi/3$ apart. Adding $2\pi k$ for $k = 3$ or more only repeats them.

4. Raising the real and imaginary parts to the power separately

Powers of a product split into powers of the factors, and the same move gets applied to a sum.

$(1+i)^{10}$	Right so far: Problem 4.
“$(1+i)^{10} = 1^{10} + i^{10}$.”	$(a+b)^n$ is not $a^n + b^n$: the binomial expansion has all the cross terms. Powers split over products, which is why polar form helps.
$(1+i)^{10} = 1 + (-1) = 0$ ✗	The right value is $32i$: $\lvert 1+i\rvert^{10} = 32$, not 0, and no power of a nonzero number is 0.

5. The modulus of a sum taken as the sum of the moduli

The modulus of a product is the product of the moduli, and the same rule seems to hold for sums.

$\lvert\omega^k\rvert = 1$ for each of the fifth roots of unity	Right so far: Problem 6.
"$\lvert 1+\omega+\omega^2+\omega^3+\omega^4\rvert = 1+1+1+1+1$."	That is only an upper bound, the triangle inequality. It is an equality only when all the nonzero terms point in the same direction.
The sum has modulus 5 ✗	The five roots point in five evenly spaced directions and cancel: the sum is 0. Likewise $\lvert 1+i\rvert + \lvert\sqrt{3}+i\rvert \approx 3.414$, but $\lvert(1+\sqrt{3})+2i\rvert \approx 3.386$.

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