Arithmetic and geometric sequences and sums

Ten problems on arithmetic and geometric sequences and their sums: a sequence from two of its terms, the sum of the multiples of 7 between 100 and 500, the first sum past 1000, a geometric sequence with a negative ratio, the derivation of the geometric sum formula, a repeating decimal as a fraction, sums in sigma notation with shifted indices, the weights of an exponential moving average and its bias correction, the sum of k x^k and the mean of a geometric distribution, and the monthly payment on a loan, with worked solutions and the mistakes of counting terms as upper minus lower, summing a geometric series with $|r| \geq 1$, putting the wrong digits over 999, bias-correcting with $1 - \beta$ instead of $1 - \beta^t$, and rounding the number of terms down.

BEFORE YOU START

An arithmetic sequence adds the same number at every step and a geometric sequence multiplies by the same number, and between them they describe straight-line growth, compound interest, repeating decimals, the decay of an exponential moving average and the first sums anyone meets in calculus. These ten problems find a sequence from two of its terms, add up long stretches of each kind, derive the geometric sum formula rather than quote it, and then use it where it matters: a repeating decimal, the bias correction in the Adam optimizer, the mean of a geometric distribution and the payment on a loan. The five mistakes at the end are the ones that give a confident wrong number: a sum in sigma notation with one term missing, a divergent series given a finite sum, the wrong digits over 999, a bias correction with the wrong power, and a number of terms rounded the wrong way.

- **Arithmetic.** With first term a_1 and common difference d, the n-th term is $a_n = a_1 + (n-1)d$, and the sum of the first n terms is $S_n = \dfrac{n(a_1 + a_n)}{2}$: the number of terms times the average of the first and last.
- **Geometric.** With first term a_1 and common ratio r, $a_n = a_1 r^{n-1}$, and for $r \neq 1$, $S_n = \dfrac{a_1(1 - r^n)}{1 - r}$. For $r = 1$, $S_n = n a_1$.
- **Infinite geometric series.** For $|r| < 1$, $r^n \to 0$ and $S_n \to \dfrac{a_1}{1 - r}$. For $|r| \geq 1$ the terms do not shrink to 0 and the series diverges (with $a_1 \neq 0$).
- **Counting.** The $n - 1$ in a_n counts steps, not terms: from a_1 to a_n is $n - 1$ steps. A sum $\sum_{k=m}^{M}$ has $M - m + 1$ terms.
- **Sigma notation.** $\sum_{k=m}^{M} c_k$ means $c_m + c_{m+1} + \cdots + c_M$. Substituting $j = k - m$ shifts the index to start at 0 without changing the sum.

Problems

1. An arithmetic sequence has $a_5 = 23$ and $a_{12} = 51$. Find d, a_1, a formula for a_n, and a_{40}. difficulty ·

2. Find the sum of all the multiples of 7 between 100 and 500. difficulty ·

3. For the arithmetic sequence $3, 7, 11, \ldots$, find S_n as a function of n, and the smallest n with $S_n > 1000$. difficulty ··

4. A geometric sequence has $a_3 = 12$ and $a_6 = -96$. Find r, a_1 and S_8. difficulty ·

5. Derive the formula $S_n = \dfrac{a_1(1 - r^n)}{1 - r}$ for $r \neq 1$. Use it to find $1 + 2 + 4 + \cdots + 2^{63}$, the grains of rice on a chessboard that doubles from one grain on the first square. difficulty ··

6. Write $2.1\overline{45} = 2.1454545\ldots$ as a fraction in lowest terms by summing a geometric series. difficulty ··

7. Evaluate both sums, and write the second with its index starting at 0: difficulty ··

$$\sum_{k=4}^{15} (2k - 5) \qquad \text{and} \qquad \sum_{k=2}^{7} 3 \cdot 2^k$$

8. An exponential moving average starts at $v_0 = 0$ and updates $v_t = \beta v_{t-1} + (1 - \beta) g_t$. Show that $v_t = (1 - \beta) \sum_{k=1}^{t} \beta^{t-k} g_k$ and that the weights add up to $1 - \beta^t$. With $\beta = 0.9$ and every $g_k = g$, find v_{10} and the bias-corrected $\hat{v}_{10} = v_{10} / (1 - \beta^{10})$. How many steps until the weights add up to more than 0.99? difficulty ···

9. Show that $\sum_{k=1}^{\infty} k x^k = \dfrac{x}{(1 - x)^2}$ for $|x| < 1$ by subtracting xT from T. Then find $\sum_{k=1}^{\infty} k / 2^k$, and the mean number of trials $\sum_{k=1}^{\infty} k p (1 - p)^{k-1}$ until the first success when each succeeds with probability $p = 0.2$. difficulty ···

10. A loan of 20 000 dollars charges 0.5% interest per month and is repaid in 60 equal monthly payments, each made at the end of a month. Find the payment and the total interest. difficulty ···

Answers

1. $d = 4$, $a_1 = 7$, $a_n = 4n + 3$, $a_{40} = 163$
2. $17\,157$
3. $S_n = n(2n + 1) = 2n^2 + n$; the smallest n is 23
4. $r = -2$, $a_1 = 3$, $S_8 = -255$
5. $S_n = \dfrac{a_1(1 - r^n)}{1 - r}$; the board holds $2^{64} - 1 \approx 1.845 \times 10^{19}$ grains
6. $\dfrac{118}{55}$
7. 168, and $\sum_{j=0}^{5} 12 \cdot 2^j = 756$
8. $v_{10} \approx 0.651\,g$, $\hat{v}_{10} = g$; the weights pass 0.99 at $t = 44$
9. $\sum k x^k = \dfrac{x}{(1 - x)^2}$; $\sum k/2^k = 2$; the mean is $1/p = 5$ trials
10. Payment ≈ 386.66 dollars; total paid $\approx 23\,199.36$, so interest $\approx 3\,199.36$ dollars

Worked solutions

Problem 1

> An arithmetic sequence has $a_5 = 23$ and $a_{12} = 51$. Find d, a_1, a formula for a_n, and a_{40}.

$7d = 51 - 23 = 28$, so $d = 4$	From term 5 to term 12 is $12 - 5 = 7$ steps.
$a_1 = 23 - 4 \cdot 4 = 7$	Walk back four steps from a_5.
$a_n = 7 + 4(n - 1) = 4n + 3$	The n-th term formula, simplified.
$a_{40} = 4 \cdot 40 + 3 = 163$	Check the formula: $a_{12} = 48 + 3 = 51$.
$d = 4$, $a_1 = 7$, $a_n = 4n + 3$, $a_{40} = 163$	An arithmetic sequence is a straight line sampled at whole numbers: $a_n = 4n + 3$ has slope $d = 4$, and any two terms fix the line.

Problem 2

> Find the sum of all the multiples of 7 between 100 and 500.

First: $7 \cdot 15 = 105$; last: $7 \cdot 71 = 497$	$100/7 \approx 14.3$ and $500/7 \approx 71.4$, so the multiples run from $7 \cdot 15$ to $7 \cdot 71$.
$n = 71 - 15 + 1 = 57$ terms	One term for each of $15, 16, \ldots, 71$.
$S_{57} = \dfrac{57(105 + 497)}{2}$	Arithmetic with $d = 7$: number of terms times the average of first and last.
$= 57 \cdot 301$	$(105 + 497)/2 = 602/2 = 301$.
$17\,157$	This is Gauss's pairing: $105 + 497$, $112 + 490$, and so on all equal 602. With an odd number of terms the middle one, $301 = 7 \cdot 43$, pairs with itself, and the formula still holds.

Problem 3

> For the arithmetic sequence $3, 7, 11, \ldots$, find S_n as a function of n, and the smallest n with $S_n > 1000$.

$a_1 = 3$, $d = 4$, $a_n = 3 + 4(n-1) = 4n - 1$	The common difference is $7 - 3 = 4$.
$S_n = \dfrac{n(3 + 4n - 1)}{2} = n(2n + 1)$	First plus last is $4n + 2$; halve it.
$2n^2 + n - 1000 = 0$ at $n = \dfrac{-1 + \sqrt{8001}}{4}$	The quadratic formula, positive root.
$n \approx 22.11$	$\sqrt{8001} \approx 89.45$.
$S_{22} = 22 \cdot 45 = 990$; $S_{23} = 23 \cdot 47 = 1081$	S_n increases with n, so check the whole numbers on each side of the root.
$S_n = n(2n+1) = 2n^2 + n$; **the smallest n is 23**	The sum of an arithmetic sequence is quadratic in n, as the area under a line is quadratic in its length. The root 22.11 must be rounded up: 22 terms are not enough (Mistake 5).

Problem 4

> A geometric sequence has $a_3 = 12$ and $a_6 = -96$. Find r, a_1 and S_8.

$r^3 = \dfrac{a_6}{a_3} = -8$, so $r = -2$	Three steps of the ratio separate term 3 from term 6; a cube root keeps the sign.
$a_1 = \dfrac{12}{(-2)^2} = 3$	$a_3 = a_1 r^2$.
$S_8 = \dfrac{3(1 - (-2)^8)}{1 - (-2)}$	The sum formula with $n = 8$.
$= \dfrac{3(1 - 256)}{3} = -255$	$(-2)^8 = 256$, and $1 - (-2) = 3$.

$r = -2$, $a_1 = 3$, $S_8 = -255$	Check by adding: $3 - 6 + 12 - 24 + 48 - 96 + 192 - 384 = -255$. With an even number of terms and $r < -1$, the last, negative term is the largest in size, so the sum is negative.

Problem 5

> Derive the formula $S_n = \dfrac{a_1(1 - r^n)}{1 - r}$ for $r \neq 1$. Use it to find $1 + 2 + 4 + \cdots + 2^{63}$, the grains of rice on a chessboard that doubles from one grain on the first square.

$S_n = a_1 + a_1 r + \cdots + a_1 r^{n-1}$	The n terms, from r^0 to r^{n-1}.
$rS_n = a_1 r + a_1 r^2 + \cdots + a_1 r^n$	Multiply every term by r: the list shifts one place.
$S_n - rS_n = a_1 - a_1 r^n$	Every term but the first of S_n and the last of rS_n cancels.
$S_n = \dfrac{a_1(1 - r^n)}{1 - r}$	Factor out S_n and divide by $1 - r$, which is allowed because $r \neq 1$.
Chessboard: $a_1 = 1$, $r = 2$, $n = 64$, so $S_{64} = \dfrac{1 - 2^{64}}{1 - 2} = 2^{64} - 1$	64 squares hold 2^0 to 2^{63}.
$S_n = \dfrac{a_1(1 - r^n)}{1 - r}$; **the board holds** $2^{64} - 1 \approx 1.845 \times 10^{19}$ **grains**	The same sum is the largest unsigned 64 -bit integer, 18 446 744 073 709 551 615: in binary it is sixty-four 1s. For $r > 1$ the form $a_1(r^n - 1)/(r - 1)$ keeps both parts positive.

Problem 6

> Write $2.1\overline{45} = 2.1454545\ldots$ as a fraction in lowest terms by summing a geometric series.

$2.1\overline{45} = 2.1 + 0.045 + 0.00045 + \ldots$	Split off the part that does not repeat; each repeat of 45 is two places further right.

The tail is geometric with $a_1 = 0.045$ and $r = 0.01$	Each term is $\frac{1}{100}$ of the one before.
$\text{Tail} = \dfrac{0.045}{1 - 0.01} = \dfrac{0.045}{0.99}$	$\lvert r \rvert < 1$, so the infinite sum is $a_1/(1 - r)$.
$= \dfrac{45}{990} = \dfrac{1}{22}$	Multiply top and bottom by 1000, then cancel 45.
$\dfrac{21}{10} + \dfrac{1}{22} = \dfrac{231 + 5}{110} = \dfrac{236}{110}$	Common denominator 110.
$\dfrac{118}{55}$	Check: $118/55 = 2.14545\ldots$. Every repeating decimal is a geometric series with ratio 10^{-p} for a block of p digits, which is why every repeating decimal is a fraction.

Problem 7

> Evaluate both sums, and write the second with its index starting at 0:
>
> $$\sum_{k=4}^{15} (2k - 5) \qquad \text{and} \qquad \sum_{k=2}^{7} 3 \cdot 2^k$$

First sum: $15 - 4 + 1 = 12$ terms, from $2 \cdot 4 - 5 = 3$ to $2 \cdot 15 - 5 = 25$	The terms go up by 2 each time: arithmetic with $d = 2$.
$\sum_{k=4}^{15} (2k - 5) = \dfrac{12(3 + 25)}{2} = 168$	Number of terms times the average of first and last.
Second sum: $j = k - 2$ runs from 0 to 5, and $3 \cdot 2^k = 3 \cdot 2^{j+2} = 12 \cdot 2^j$	Substituting $k = j + 2$ moves the shift into the coefficient.
$\sum_{j=0}^{5} 12 \cdot 2^j = \dfrac{12(2^6 - 1)}{2 - 1}$	Geometric with $a_1 = 12$, $r = 2$, and 6 terms.
$= 12 \cdot 63 = 756$	Check: $12 + 24 + 48 + 96 + 192 + 384 = 756$.

168, **and** $\sum_{j=0}^{5} 12 \cdot 2^j = 756$	Before applying a formula, read off three things from the sigma: the first term, the ratio or difference, and the number of terms, upper minus lower plus one.

Problem 8

> An exponential moving average starts at $v_0 = 0$ and updates $v_t = \beta v_{t-1} + (1-\beta) g_t$. Show that $v_t = (1-\beta)\sum_{k=1}^{t} \beta^{t-k} g_k$ and that the weights add up to $1 - \beta^t$. With $\beta = 0.9$ and every $g_k = g$, find v_{10} and the bias-corrected $\hat{v}_{10} = v_{10}/(1-\beta^{10})$. How many steps until the weights add up to more than 0.99?

$v_1 = (1-\beta)g_1$, $v_2 = (1-\beta)(\beta g_1 + g_2)$, and so on	Unroll the recursion from $v_0 = 0$: each step multiplies the old terms by β and adds a new one.
$v_t = (1-\beta)\sum_{k=1}^{t} \beta^{t-k} g_k$	By induction: βv_{t-1} raises every power by one, and $(1-\beta)g_t$ is the $k = t$ term with β^0.
Weights: $(1-\beta)(1 + \beta + \cdots + \beta^{t-1})$	The powers β^{t-k} for $k = t, t-1, \ldots, 1$, a geometric sum of t terms.
$= (1-\beta)\cdot \dfrac{1-\beta^t}{1-\beta} = 1 - \beta^t$	Problem 5's formula with $a_1 = 1$ and $r = \beta$.
$v_{10} = (1 - 0.9^{10})g \approx 0.6513\, g$	With a constant input, v_t is the total weight times g; $0.9^{10} \approx 0.3487$.
$\hat{v}_{10} = \dfrac{0.6513\, g}{0.6513} = g$	Dividing by the total weight makes the weights add up to 1.
$0.9^t < 0.01$ when $t > \dfrac{\ln 0.01}{\ln 0.9} \approx 43.7$	Take logs; $\ln 0.9 < 0$ flips the inequality.
$v_{10} \approx 0.651\, g$, $\hat{v}_{10} = g$; **the weights pass** 0.99 **at** $t = 44$	Starting from 0 biases the average toward 0 by the factor $1 - \beta^t$, and Adam's bias correction divides it out. By $t = 44$ the factor is $1 - 0.9^{44} \approx 0.9903$ and the correction barely matters; with $\beta = 0.999$, as for Adam's second moment, it takes about 4600 steps.

Problem 9

> Show that $\sum_{k=1}^{\infty} kx^k = \dfrac{x}{(1-x)^2}$ for $|x| < 1$ by subtracting xT from T. Then find $\sum_{k=1}^{\infty} k/2^k$, and the mean number of trials $\sum_{k=1}^{\infty} kp(1-p)^{k-1}$ until the first success when each succeeds with probability $p = 0.2$.

$T = x + 2x^2 + 3x^3 + \dots$ and $xT = x^2 + 2x^3 + \dots$	Multiply by x: each coefficient moves one power up.
$T - xT = x + x^2 + x^3 + \dots$	Coefficients k and $k-1$ meet at each power x^k, leaving 1.
$= \dfrac{x}{1-x}$	A geometric series with $a_1 = x$, $r = x$ and $\lvert x\rvert < 1$.
$T = \dfrac{x}{(1-x)^2}$	Divide by $1 - x$. The series converges for $\lvert x\rvert < 1$, since the ratio of consecutive terms, $\frac{k+1}{k}\lvert x\rvert$, tends to $\lvert x\rvert < 1$; so T is a finite number and the subtraction is allowed.
$x = \frac{1}{2}$: $T = \dfrac{1/2}{1/4} = 2$	Partial sums: 0.5, 1, 1.375, 1.625, …
Mean: $\dfrac{p}{1-p}\sum_k k(1-p)^k = \dfrac{p}{1-p} \cdot \dfrac{1-p}{p^2} = \dfrac{1}{p}$	Take one factor $1-p$ outside, then step 4 with $x = 1-p$, $1-x = p$.
$\sum kx^k = \dfrac{x}{(1-x)^2}$; $\sum k/2^k = 2$; **the mean is** $1/p = 5$ **trials**	A sum whose terms are an arithmetic sequence times a geometric one is called arithmetico-geometric, and the subtraction that proves the geometric formula reduces it to a geometric sum. The same answer comes from differentiating $\sum x^k = 1/(1-x)$ and multiplying by x.

Problem 10

> A loan of 20 000 dollars charges 0.5% interest per month and is repaid in 60 equal monthly payments, each made at the end of a month. Find the payment and the total interest.

After n months the debt has grown to $L(1+i)^n$ with $L = 20\,000$, $i = 0.005$	Each month multiplies the balance by $1+i$.
Payment k has grown to $P(1+i)^{60-k}$ by month 60	It is made at the end of month k and earns interest for the remaining months.
$P\sum_{j=0}^{59}(1+i)^j = P\cdot\dfrac{(1+i)^{60}-1}{i}$	A geometric sum with $a_1 = 1$, $r = 1+i$ and 60 terms.
$L(1+i)^{60} = P\cdot\dfrac{(1+i)^{60}-1}{i}$	The loan is repaid when the payments, grown to month 60, equal the grown debt.
$P = \dfrac{Li}{1-(1+i)^{-60}} = \dfrac{100}{1-1.005^{-60}}$	Divide both sides by $(1+i)^{60}$; $Li = 100$.
$1.005^{60} \approx 1.34885$, so $1 - 1.005^{-60} \approx 0.25863$	One power, then its reciprocal.
Payment ≈ 386.66 dollars; total paid $\approx 23\,199.36$, so interest $\approx 3\,199.36$ dollars	$60 \times 386.656 \approx 23\,199.36$. Simple interest on the full 20 000 for five years would be 6 000; it is about half that because the balance falls as it is repaid.

Where this goes wrong

1. Counting the terms as upper minus lower

A sum $\sum_{k=4}^{15}$ looks like it covers a stretch of length $15-4$.

$\sum_{k=4}^{15}(2k-5)$ runs from 3 to 25	Right so far: Problem 7, step 1.

“There are $15 - 4 = 11$ terms.”	The subtraction counts the steps between $k = 4$ and $k = 15$, and there is one more term than there are steps.
$S = \dfrac{11(3 + 25)}{2} = 154$ ✗	The sum has $15 - 4 + 1 = 12$ terms and equals 168. A quick check is to count a short case by hand: $\sum_{k=4}^{5}$ has two terms, not one.

2. Summing a geometric series with |r| ≥ 1

The formula $a_1/(1 - r)$ gives a number for any $r \neq 1$, and nothing in the arithmetic stops at the condition.

$\sum_{k=0}^{\infty} \left(\frac{3}{2}\right)^k$ is geometric with $a_1 = 1$ and $r = \frac{3}{2}$	Right so far.
“The infinite sum is first term over $1 - r$. ”	That holds only when $\lvert r \rvert < 1$, because the formula is the limit of $a_1(1 - r^n)/(1 - r)$ and needs $r^n \to 0$.
$\sum_{k=0}^{\infty} \left(\frac{3}{2}\right)^k = \dfrac{1}{1 - 3/2} = -2$ ✗	A sum of positive terms cannot be negative. Here r^n grows without bound, the partial sums $1, 2.5, 4.75, \ldots$ do too, and the series diverges.

3. Putting the wrong digits over 999

The rule that a repeating block of p digits goes over p nines is remembered without the condition that the block starts right after the decimal point.

$2.1\overline{45} = 2.1454545\ldots$	Right so far: Problem 6.
“The digits after the point are 145, three of them, so the fraction is $145/999$.”	Only 45 repeats; the 1 is written once. Putting it inside the block repeats it too.
$2.1\overline{45} = 2 + \dfrac{145}{999}$ ✗	That is $2.145145\ldots$, not $2.14545\ldots$. Split off the part that does not repeat first: $2.1 + \dfrac{45}{990} = \dfrac{118}{55}$.

4. Bias correction with 1 – β instead of 1 – β^t

The update has a $1-\beta$ in it, and dividing by the same factor looks like the way to undo it.

$v_{10} = (1-0.9^{10})g \approx 0.6513\,g$	Right so far: Problem 8, step 5.
"The average is scaled by $1-\beta$, so divide by $1-\beta$."	The $1-\beta$ in the update is already balanced by the sum of the powers of β; what is missing is the weight on the steps before $t=1$, which is β^t.
$\hat{v}_{10} = \dfrac{v_{10}}{1-0.9} \approx 6.51\,g$ ✗	That is more than six times the true average. The total weight is $1-\beta^t$ (Problem 8, step 4), and dividing by it gives $\hat{v}_{10} = g$. With $1-\beta$ the estimate tends to $10g$ instead of g as t grows.

5. Rounding the number of terms down

The quadratic formula gives a decimal, and rounding to the nearest whole number is the usual reflex.

$2n^2 + n = 1000$ at $n \approx 22.11$	Right so far: Problem 3, step 4.
"Round 22.11 to 22."	The question asks for $S_n > 1000$, and S_n increases, so n must be at least the root.
$n = 22$ ✗	$S_{22} = 990$, still short of 1000. The smallest n is 23, with $S_{23} = 1081$. When the answer must exceed a threshold, round up and check both neighbors.

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