Area between curves

Ten areas between curves: top minus bottom between the intersection points, regions that cross in the middle, a region integrated in y, a log and an integration by parts, a horizontal line that halves an area, and the area between two probability densities, with worked solutions and the mistakes that give a negative area or a total of zero.

BEFORE YOU START

The area between two curves is the integral of the gap between them, and the same integral measures how far apart two functions are: the area between two probability densities is their L^1 distance, and half of it is the total variation distance. The work goes wrong in a few places: the curves subtracted in the wrong order, an integral run straight across a point where the curves cross, limits taken from the picture instead of from the intersections, and a region that is one integral in y but two in x. These ten problems cover each of them.

- If $f(x) \geq g(x)$ for every x in $[a, b]$, the area between the graphs of f and g from $x = a$ to $x = b$ is $\displaystyle\int_a^b \big(f(x) - g(x)\big)\,dx$: top minus bottom.
- If the curves cross inside $[a, b]$, which one is on top changes. The area is $\displaystyle\int_a^b |f(x) - g(x)|\,dx$, computed by splitting the interval at each crossing and taking top minus bottom on each piece. Without the split, the pieces where g is on top count as negative and cancel area, which is the net signed area, not the area.
- The limits of a region enclosed by two curves are where they meet: solve $f(x) = g(x)$.
- If a region is bounded by curves given as x in terms of y, or if its top or bottom changes formula partway across, integrate in y instead: area $= \displaystyle\int_c^d \big(x_{\text{right}}(y) - x_{\text{left}}(y)\big)\,dy$, right minus left.
- An area is never negative. A negative result means the subtraction was the wrong way round.
- Integrals used: $\int x^n\,dx = \frac{x^{n+1}}{n+1}$ for $n \neq -1$, $\int \frac{1}{x}\,dx = \ln|x|$, $\int e^x\,dx = e^x$, $\int \sin x\,dx = -\cos x$, $\int \cos x\,dx = \sin x$, and $\int \ln x\,dx = x\ln x - x$ (by parts).

Builds on: Integration: substitution and parts

Problems

1. Find the area of the region enclosed by $y = x$ and $y = x^2$. difficulty ·

2. Find the area of the region enclosed by $y = 4 - x^2$ and $y = x + 2$. difficulty ·

3. Find the total area of the regions enclosed by $y = x^3$ and $y = x$. difficulty ··

4. Find the area between $y = \sin x$ and $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$. difficulty ··

5. Find the area of the region bounded by $y = \sqrt{x}$, $y = x - 2$ and the x-axis, first by integrating in y, then in x. difficulty ··

6. Find the area of the region enclosed by $y = \dfrac{1}{x}$ and $y = \dfrac{5}{2} - x$. difficulty ··

7. The region between $y = x^2$ and $y = 4$ is cut by a horizontal line $y = c$. Find c so that the two parts have equal area. difficulty ···

8. Two probability densities on $[0, 1]$ are $p(x) = 2x$ and $q(x) = 1$. Find the area between their graphs, and the total variation distance $\frac{1}{2}\int_0^1 |p(x) - q(x)|\,dx$. difficulty ···

9. Find the area between $y = \ln x$ and the line through $(1, 0)$ and $(e, 1)$. difficulty ···

10. Find the total area of the regions between $y = \sin x$ and $y = \sin 2x$ on $[0, \pi]$. difficulty ···

Answers

1. $\frac{1}{6}$
2. $\frac{9}{2}$
3. $\frac{1}{2}$
4. $2\sqrt{2} - 2 \approx 0.83$
5. $\frac{10}{3}$
6. $\frac{15}{8} - 2\ln 2 \approx 0.489$
7. $c = 4^{2/3} = 2^{4/3} \approx 2.52$
8. Area $\frac{1}{2}$; total variation distance $\frac{1}{4}$
9. $1 - \dfrac{e-1}{2} = \dfrac{3-e}{2} \approx 0.141$
10. $\frac{5}{2}$

Worked solutions

Problem 1

> Find the area of the region enclosed by $y = x$ and $y = x^2$.

$x = x^2$ gives $x(x-1) = 0$, so $x = 0$ or $x = 1$.	The region is enclosed by the two curves, so its limits are where they meet.
On $(0, 1)$, $x > x^2$.	Test a point: at $x = \frac{1}{2}$, $\frac{1}{2} > \frac{1}{4}$. The line is on top.
$\text{Area} = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$	Top minus bottom.
$\frac{1}{6}$	$\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

Problem 2

> Find the area of the region enclosed by $y = 4 - x^2$ and $y = x + 2$.

$4 - x^2 = x + 2$ gives $x^2 + x - 2 = (x + 2)(x - 1) = 0$, so $x = -2$ or $x = 1$.	The intersections are the limits.
On $(-2, 1)$, $4 - x^2 > x + 2$.	At $x = 0$: $4 > 2$. The parabola is on top.
$\text{Area} = \int_{-2}^1 \big((4 - x^2) - (x + 2)\big)\,dx = \int_{-2}^1 (2 - x - x^2)\,dx$	Top minus bottom; the parentheses around $x + 2$ make both of its terms subtracted.
$\left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^1 = \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right) = \frac{7}{6} + \frac{10}{3}$	The antiderivative at the upper limit minus at the lower one.
$\frac{9}{2}$	$\frac{7}{6} + \frac{20}{6} = \frac{27}{6} = \frac{9}{2}$.

Problem 3

> Find the total area of the regions enclosed by $y = x^3$ and $y = x$.

$x^3 = x$ gives $x(x-1)(x+1) = 0$, so $x = -1, 0, 1$.	Three intersections: two enclosed regions, $[-1, 0]$ and $[0, 1]$.
On $(-1, 0)$, $x^3 > x$; on $(0, 1)$, $x > x^3$.	At $x = -\frac{1}{2}$: $-\frac{1}{8} > -\frac{1}{2}$. At $x = \frac{1}{2}$: $\frac{1}{2} > \frac{1}{8}$. The top curve changes at 0.
$\int_{-1}^{0} (x^3 - x)\,dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} = 0 - \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{1}{4}$	Top minus bottom on the left piece.
$\int_{0}^{1} (x - x^3)\,dx = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$	Top minus bottom on the right piece. Both curves are odd functions, so the two pieces are mirror images and the areas match.
$\frac{1}{2}$	$\frac{1}{4} + \frac{1}{4}$. A single integral $\int_{-1}^{1} (x - x^3)\,dx$ gives 0, because its two halves cancel.

Problem 4

> Find the area between $y = \sin x$ and $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$.

$\sin x = \cos x$ on $[0, \frac{\pi}{2}]$ at $x = \frac{\pi}{4}$ only.	$\tan x = 1$; the curves cross inside the interval, so it must be split there.
On $(0, \frac{\pi}{4})$, $\cos x > \sin x$; on $(\frac{\pi}{4}, \frac{\pi}{2})$, $\sin x > \cos x$.	At 0, $\cos 0 = 1 > 0 = \sin 0$; at $\frac{\pi}{2}$ the order is reversed.
$\int_{0}^{\pi/4} (\cos x - \sin x)\,dx = \Big[\sin x + \cos x\Big]_{0}^{\pi/4} = \sqrt{2} - 1$	$\sin \frac{\pi}{4} + \cos \frac{\pi}{4} = \sqrt{2}$ and $\sin 0 + \cos 0 = 1$.

$\int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx = \Big[-\cos x - \sin x \Big]_{\pi/4}^{\pi/2} = -1 - (-\sqrt{2}) = \sqrt{2} - 1$	Top minus bottom on the second piece. The picture is symmetric about $x = \frac{\pi}{4}$, so the pieces are equal.
$2\sqrt{2} - 2 \approx 0.83$	The sum of the two pieces. Integrating $\cos x - \sin x$ over the whole interval gives $(1 + 0) - (0 + 1) = 0$ instead (mistake 2).

Problem 5

> Find the area of the region bounded by $y = \sqrt{x}$, $y = x - 2$ and the x-axis, first by integrating in y, then in x.

In y: $x = y^2$ is the left boundary and $x = y + 2$ the right one, for $0 \le y \le 2$.	Solve each curve for x. They meet where $y^2 = y + 2$, at $y = 2$ ($y = -1$ is below the x-axis, outside the region), and the x-axis is $y = 0$.
$\text{Area} = \int_0^2 \big((y + 2) - y^2\big)\,dy = \Big[\frac{y^2}{2} + 2y - \frac{y^3}{3} \Big]_0^2 = 2 + 4 - \frac{8}{3}$	Right minus left, one integral.
In x: the bottom is $y = 0$ for $0 \le x \le 2$ and $y = x - 2$ for $2 \le x \le 4$; the top is $\sqrt{x}$ throughout.	The line crosses the x-axis at $x = 2$ and meets $\sqrt{x}$ at $x = 4$ (where $\sqrt{4} = 2 = 4 - 2$), so the bottom changes formula at $x = 2$.
$\int_0^2 \sqrt{x}\,dx + \int_2^4 \big(\sqrt{x} - x + 2\big)\,dx = \frac{2}{3}2^{3/2} + \Big[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x \Big]_2^4$	Two integrals, one for each bottom.
$= \frac{4\sqrt{2}}{3} + \Big(\frac{16}{3} - 8 + 8 \Big) - \Big(\frac{4\sqrt{2}}{3} - 2 + 4 \Big) = \frac{16}{3} - 2$	The $\frac{4\sqrt{2}}{3}$ terms cancel.
$\frac{10}{3}$	Both routes: $6 - \frac{8}{3} = \frac{10}{3}$ and $\frac{16}{3} - 2 = \frac{10}{3}$. In y it is one integral of a polynomial.

Problem 6

> Find the area of the region enclosed by $y = \dfrac{1}{x}$ and $y = \dfrac{5}{2} - x$.

$\frac{1}{x} = \frac{5}{2} - x$ gives $2 = 5x - 2x^2$, so $2x^2 - 5x + 2 = (2x - 1)(x - 2) = 0$.	Multiply both sides by $2x$, which is not 0 at an intersection.
$x = \frac{1}{2}$ or $x = 2$, and on $(\frac{1}{2}, 2)$ the line is on top.	At $x = 1$: $\frac{5}{2} - 1 = \frac{3}{2} > 1$.
$\text{Area} = \displaystyle\int_{1/2}^{2} \left(\frac{5}{2} - x - \frac{1}{x}\right) dx = \left[\frac{5}{2}x - \frac{x^2}{2} - \ln x\right]_{1/2}^{2}$	$x > 0$ on the interval, so $\ln\lvert x\rvert = \ln x$.
$= (5 - 2 - \ln 2) - \left(\frac{5}{4} - \frac{1}{8} - \ln \frac{1}{2}\right) = (3 - \ln 2) - \left(\frac{9}{8} + \ln 2\right)$	$\ln \frac{1}{2} = -\ln 2$, so $-\ln \frac{1}{2} = +\ln 2$.
$\frac{15}{8} - 2\ln 2 \approx 0.489$	$3 - \frac{9}{8} = \frac{15}{8}$, and the two $\ln 2$ terms add. Sanity check: the gap is at most $\frac{1}{2}$ (at $x = 1$) over an interval of length $\frac{3}{2}$, so the area is below $\frac{3}{4}$.

Problem 7

> The region between $y = x^2$ and $y = 4$ is cut by a horizontal line $y = c$. Find c so that the two parts have equal area.

$\text{Total area} = \displaystyle\int_{-2}^{2} (4 - x^2)\, dx = 16 - \frac{16}{3} = \frac{32}{3}$.	The curves meet at $x = \pm 2$; top minus bottom.
At height y the region runs from $x = -\sqrt{y}$ to $x = \sqrt{y}$, a width of $2\sqrt{y}$.	The cut is horizontal, so slice horizontally: integrate in y.
$\text{Area below } y = c\text{: } \displaystyle\int_{0}^{c} 2\sqrt{y}\, dy = \frac{4}{3}c^{3/2}$.	Right minus left is $2\sqrt{y}$, from the bottom of the region, $y = 0$, up to the cut.

$\frac{4}{3}c^{3/2} = \frac{16}{3}$, so $c^{3/2} = 4$.	Half of $\frac{32}{3}$. As a check, $c = 4$ gives $\frac{4}{3} \cdot 8 = \frac{32}{3}$, the whole region.
$c = 4^{2/3} = 2^{4/3} \approx 2.52$	Raise both sides to the power $\frac{2}{3}$. The cut is above the halfway height $y = 2$, because the region is narrow at the bottom (mistake 4).

Problem 8

> Two probability densities on $[0, 1]$ are $p(x) = 2x$ and $q(x) = 1$. Find the area between their graphs, and the total variation distance $\frac{1}{2}\int_0^1 |p(x) - q(x)|\,dx$.

$\displaystyle\int_0^1 \big(p(x) - q(x)\big)\,dx = 1 - 1 = 0.$	Every density integrates to 1, so the signed integral of the difference of two densities is always 0. It says nothing about how different they are.
$2x = 1$ at $x = \frac{1}{2}$; q is on top on $(0, \frac{1}{2})$ and p on $(\frac{1}{2}, 1)$.	The crossing splits the interval.
$\displaystyle\int_0^{1/2} (1 - 2x)\,dx = \Big[x - x^2\Big]_0^{1/2} = \frac{1}{4}$ and $\displaystyle\int_{1/2}^1 (2x - 1)\,dx = \Big[x^2 - x\Big]_{1/2}^1 = 0 - \big(-\frac{1}{4}\big) = \frac{1}{4}$	Top minus bottom on each piece. The two pieces are equal, as step 1 forces: their difference is the signed integral, 0.
Area $\frac{1}{2}$; total variation distance $\frac{1}{4}$	The area is $\frac{1}{4} + \frac{1}{4}$, and the distance is half of it. It is also the largest difference in probability the two densities give any event: $P(x > \frac{1}{2})$ is $\frac{3}{4}$ under p and $\frac{1}{2}$ under q.

Problem 9

> Find the area between $y = \ln x$ and the line through $(1, 0)$ and $(e, 1)$.

The line is $y = \dfrac{x - 1}{e - 1}$, and it meets $\ln x$ at $x = 1$ and $x = e$.	Slope $\frac{1-0}{e-1}$ through $(1, 0)$. Both points are on $y = \ln x$, since $\ln 1 = 0$ and $\ln e = 1$.

On $(1, e)$, $\ln x$ is above the line.	$\ln x$ is concave, so it lies above every chord. At $x = 2$: $\ln 2 \approx 0.693 > \frac{1}{e-1} \approx 0.582$.
$\int_1^e \ln x \, dx = \Big[x \ln x - x\Big]_1^e = (e - e) - (0 - 1) = 1$	By parts, with $u = \ln x$ and $dv = dx$.
$\int_1^e \frac{x-1}{e-1} \, dx = \frac{1}{e-1} \cdot \frac{(e-1)^2}{2} = \frac{e-1}{2}$	$\int_1^e (x-1)\,dx = \Big[\frac{(x-1)^2}{2}\Big]_1^e$. Geometrically, a triangle with base $e-1$ and height 1.
$1 - \frac{e-1}{2} = \frac{3-e}{2} \approx 0.141$	Top minus bottom: step 3 minus step 4.

Problem 10

Find the total area of the regions between $y = \sin x$ and $y = \sin 2x$ on $[0, \pi]$.

$\sin 2x = \sin x$ gives $2 \sin x \cos x - \sin x = \sin x \, (2 \cos x - 1) = 0$.	The double-angle formula, then factor. Dividing by $\sin x$ would lose the crossings at 0 and π.
On $[0, \pi]$: $\sin x = 0$ at $x = 0, \pi$, and $\cos x = \frac{1}{2}$ at $x = \frac{\pi}{3}$.	So the curves cross inside the interval at $\frac{\pi}{3}$.
On $(0, \frac{\pi}{3})$, $\sin 2x > \sin x$; on $(\frac{\pi}{3}, \pi)$, $\sin x > \sin 2x$.	$\sin 2x - \sin x = \sin x \, (2 \cos x - 1)$, and $\sin x > 0$ on $(0, \pi)$, so the sign is that of $2 \cos x - 1$: positive before $\frac{\pi}{3}$, negative after.
$\int_0^{\pi/3} (\sin 2x - \sin x) \, dx = \Big[- \frac{\cos 2x}{2} + \cos x\Big]_0^{\pi/3} = \Big(\frac{1}{4} + \frac{1}{2}\Big) - \Big(-\frac{1}{2} + 1\Big) = \frac{1}{4}$	$\cos \frac{2\pi}{3} = -\frac{1}{2}$ and $\cos \frac{\pi}{3} = \frac{1}{2}$.
$\int_{\pi/3}^{\pi} (\sin x - \sin 2x) \, dx = \Big[- \cos x + \frac{\cos 2x}{2}\Big]_{\pi/3}^{\pi} = \Big(1 + \frac{1}{2}\Big) - \Big(-\frac{1}{2} - \frac{1}{4}\Big) = \frac{9}{4}$	$\cos \pi = -1$ and $\cos 2\pi = 1$.

$\frac{5}{2}$	$\frac{1}{4} + \frac{9}{4}$. The single integral $\int_0^{\pi} (\sin x - \sin 2x)\,dx$ is $2 - 0 = 2$, short by twice the first piece.

Where this goes wrong

1. Bottom minus top

The formula is remembered as "one curve minus the other", and the order is left to chance.

$y = 4 - x^2$ and $y = x + 2$ meet at $x = -2$ and $x = 1$	Right so far: Problem 2, step 1.
"Subtract the curves in the order they were given in the problem."	The habit that causes the mistake: not checking which curve is on top.
Area $= \int_{-2}^{1} \big((x + 2) - (4 - x^2)\big)\,dx = -\frac{9}{2}$ ✗	The line is below the parabola on $(-2, 1)$, so this integrates bottom minus top and every slice counts as negative. An area cannot be negative. Test one point between the limits to see which curve is on top, then subtract the bottom from it: the area is $\frac{9}{2}$.

2. One integral across a crossing

When the limits are given, it is tempting to subtract once and integrate straight across them.

Area between $\sin x$ and $\cos x$ on $[0, \frac{\pi}{2}]$	Right so far: Problem 4.
"$\cos x$ starts on top, so integrate $\cos x - \sin x$ from 0 to $\frac{\pi}{2}$."	The habit that causes the mistake: checking which curve is on top at one end only.
$\int_0^{\pi/2} (\cos x - \sin x)\,dx = \Big[\sin x + \cos x\Big]_0^{\pi/2} = 1 - 1 = 0$ ✗	The curves cross at $\frac{\pi}{4}$, and after it $\cos x - \sin x$ is negative, so the second region cancels the first. Zero area between two curves that are visibly apart is the giveaway. Split at the crossing: the area is $2(\sqrt{2} - 1) \approx 0.83$.

3. Missing a boundary

A region with three boundaries gets treated as if it had two.

Region bounded by $y = \sqrt{x}$, $y = x - 2$ and the x-axis	Right so far: Problem 5.
"$\sqrt{x}$ is on top and $x - 2$ is on the bottom, and they meet at $x = 4$."	The habit that causes the mistake: looking only at the two named curves.
Area $= \int_0^4 \left(\sqrt{x} - (x-2)\right)\,dx = \frac{16}{3} - 8 + 8 = \frac{16}{3}$ ✗	On $[0, 2]$ the line is below the x-axis, and the region stops at the axis, so this adds the triangle between the axis and the line, with base 2 and height 2, area 2. The correct area is $\frac{16}{3} - 2 = \frac{10}{3}$. Sketch the region before choosing a top and a bottom.

4. Halving the height instead of the area

"Split the region in half" sounds like "cut it at half its height".

Region between $y = x^2$ and $y = 4$, area $\frac{32}{3}$, cut by $y = c$	Right so far: Problem 7, step 1.
"The region runs from $y = 0$ to $y = 4$, so the cut is at the middle."	The habit that causes the mistake: treating the region as if it had the same width at every height.
$c = 2$ ✗	The area below $y = 2$ is $\frac{4}{3} \cdot 2^{3/2} = \frac{8\sqrt{2}}{3} \approx 3.77$, less than half of $\frac{32}{3} \approx 10.67$, which is about 5.33. The region is narrow at the bottom, where its width $2\sqrt{y}$ is small, so the cut must be higher: $c = 2^{4/3} \approx 2.52$.

5. Losing the sign of ln ½

The lower limit of a log integral is where signs get dropped.

$\left[\frac{5}{2}x - \frac{x^2}{2} - \ln x\right]_{1/2}^{2}$	Right so far: Problem 6, step 3.

"At $x = \frac{1}{2}$: $\frac{5}{4} - \frac{1}{8} - \ln 2$."	The habit that causes the mistake: writing $\ln \frac{1}{2}$ as $\ln 2$, as if the log of a fraction were the log of its denominator.
$\text{Area} = (3 - \ln 2) - \left(\frac{9}{8} - \ln 2\right) = \frac{15}{8}$ ✗	$\ln \frac{1}{2} = -\ln 2$, so the lower limit contributes $\frac{9}{8} + \ln 2$ and the $\ln 2$ terms add instead of canceling. $\frac{15}{8} = 1.875$ cannot be right: the gap between the curves is at most $\frac{1}{2}$ over a width of $\frac{3}{2}$, so the area is under 0.75. The area is $\frac{15}{8} - 2\ln 2 \approx 0.489$.

More at eulerpad.app. This is one of 82 free practice sets with worked solutions at eulerpad.app/practice, alongside guides to the topics behind them.

EulerPad, the app, reads your handwritten math on iPad, iPhone and Android and checks it line by line. It's free to start.

This set is also Chapter 17 of *Calculus I & II*, a 250-problem workbook with worked solutions: eulerpad.app/workbooks

eulerpad.app/practice/area-between-curves