# Arc length and surface area of revolution

Ten problems on the length of a curve and the area of a surface of revolution: a line checked against the distance formula, y = x^(3/2), the catenary, a curve built so that 1 + (y′)² is a perfect square, arc length in y, a circular arc that gives back the angle, y = x² by trig substitution, the sphere's 4πr², √x turned about the x-axis and x² turned about the y-axis, with worked solutions and the mistakes of taking the square root term by term, leaving out the slant factor and measuring the radius from the wrong axis.

**BEFORE YOU START**

The length of a curve is found the same way as an area: cut it into tiny pieces, approximate each piece by something simple, and add. For a curve, the simple thing is a short straight segment, and Pythagoras gives its length. Turning the curve around an axis sweeps each segment into a thin band, and adding the bands gives the area of the surface. The formulas are short, but the square root in them rarely has an antiderivative, so the problems that can be done by hand are the ones where $1 + (y')^2$ turns out to be a perfect square, or where a substitution clears it. The work goes wrong in a few places: the square root split term by term, the derivative taken in the wrong variable, the slant factor left out of a surface area, the radius measured from the wrong axis, and the disk formula used for a surface. These ten problems cover each of them.

- **Arc length.** For $y = f(x)$ on $[a, b]$, $L = \displaystyle\int_a^b \sqrt{1 + f'(x)^2}\,dx$. For $x = g(y)$ on $[c, d]$, $L = \displaystyle\int_c^d \sqrt{1 + g'(y)^2}\,dy$, with the derivative and the limits both in $y$.
- **Where it comes from.** A short piece of curve is nearly straight, with length $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (dy/dx)^2}\,dx$.
- **Perfect squares.** If $f'(x) = A - B$ with $4AB = 1$, then $1 + f'^2 = (A + B)^2$ and $\sqrt{1 + f'^2} = A + B$. Curves like $\frac{x^3}{6} + \frac{1}{2x}$ are built this way.
- **Surface area.** Turning a curve about an axis gives $S = \displaystyle\int 2\pi r\,ds$, where $r$ is the distance from the curve to the axis. About the $x$-axis $r = |y|$; about the $y$-axis $r = |x|$.
- **Checks.** A sphere of radius $r$ has area $4\pi r^2$, and an arc of a circle of radius $r$ through angle $\theta$ has length $r\theta$. An arc length is never shorter than the straight chord between its ends, and a surface area has units of length squared.

Builds on: Integration: substitution and parts, Volumes of revolution: disks, washers and shells

## Problems

**1.** Find the length of $y = 2x + 1$ from $x = 0$ to $x = 3$ with the arc length integral, and check it with the distance formula. difficulty ·

**2.** Find the length of $y = x^{3/2}$ from $x = 0$ to $x = 4$. difficulty ·

**3.** The hanging-cable curve $y = \cosh x = \dfrac{e^x + e^{-x}}{2}$. Find its length from $x = 0$ to $x = \ln 2$. difficulty ··

**4.** Find the length of $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$ from $x = 1$ to $x = 2$. difficulty ··

**5.** Find the length of the curve $x = \frac{2}{3}(y - 1)^{3/2}$ from $y = 1$ to $y = 4$. difficulty ··

**6.** Find the length of $y = \sqrt{1 - x^2}$ from $x = 0$ to $x = \frac{1}{2}$, and explain the answer with a picture. difficulty ··

**7.** Find the length of the parabola $y = x^2$ from $x = 0$ to $x = 1$. difficulty ···

**8.** Turn the semicircle $y = \sqrt{r^2 - x^2}$, $-r \le x \le r$, about the $x$-axis, and find the surface area of the sphere. difficulty ·

**9.** Find the area of the surface made by turning $y = \sqrt{x}$, $0 \le x \le 2$, about the $x$-axis. difficulty ··

**10.** Find the area of the surface made by turning $y = x^2$, $0 \le x \le 1$, about the $y$-axis. difficulty ···

## Answers

1. $3\sqrt{5} \approx 6.708$
2. $\dfrac{8}{27}\left(10\sqrt{10} - 1\right) \approx 9.073$
3. $\sinh(\ln 2) = \dfrac{2 - \frac{1}{2}}{2} = \dfrac{3}{4}$
4. $\dfrac{17}{12} \approx 1.417$
5. $\dfrac{2}{3}(8 - 1) = \dfrac{14}{3} \approx 4.667$
6. $\dfrac{\pi}{6} \approx 0.5236$
7. $L = \dfrac{\sqrt{5}}{2} + \dfrac{1}{4}\ln\left(2 + \sqrt{5}\right) \approx 1.4789$
8. $S = 4\pi r^2$
9. $S = \dfrac{13\pi}{3} \approx 13.614$
10. $S = \dfrac{\pi}{6}\left(5\sqrt{5} - 1\right) \approx 5.330$

## Worked solutions

### Problem 1

> Find the length of $y = 2x + 1$ from $x = 0$ to $x = 3$ with the arc length integral, and check it with the distance formula.

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|---|---|
| $y' = 2$, so $\sqrt{1 + y'^2} = \sqrt{5}$ | A line has constant slope. |
| $L = \displaystyle\int_0^3 \sqrt{5}\,dx = 3\sqrt{5}$ | The integrand is a constant. |
| The ends are $(0, 1)$ and $(3, 7)$, at distance $\sqrt{3^2 + 6^2} = \sqrt{45} = 3\sqrt{5}$ | The two agree. |
| $3\sqrt{5} \approx 6.708$ | For a line, the arc length formula is the distance formula. |

### Problem 2

> Find the length of $y = x^{3/2}$ from $x = 0$ to $x = 4$.

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|---|---|
| $y' = \frac{3}{2}x^{1/2}$, so $1 + y'^2 = 1 + \frac{9}{4}x$ | Square the derivative. |
| $L = \displaystyle\int_0^4 \left(1 + \frac{9}{4}x\right)^{1/2} dx$ | Substitute $u = 1 + \frac{9}{4}x$, $du = \frac{9}{4}\,dx$; $u$ runs from 1 to 10. |
| $= \dfrac{4}{9} \cdot \dfrac{2}{3}\left[u^{3/2}\right]_1^{10} = \dfrac{8}{27}\left(10^{3/2} - 1\right)$ | The power rule in $u$. |
| $\dfrac{8}{27}\left(10\sqrt{10} - 1\right) \approx 9.073$ | The chord from $(0, 0)$ to $(4, 8)$ has length $\sqrt{80} \approx 8.944$, a little shorter. |

### Problem 3

> The hanging-cable curve $y = \cosh x = \dfrac{e^x + e^{-x}}{2}$. Find its length from $x = 0$ to $x = \ln 2$.

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| $y' = \sinh x = \dfrac{e^x - e^{-x}}{2}$ | Differentiate each exponential. |

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| $1 + \sinh^2 x = \cosh^2 x$ | Expand: $1 + \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{e^{2x} + 2 + e^{-2x}}{4}$, a perfect square. |
| $L = \int_0^{\ln 2} \cosh x \, dx = \sinh(\ln 2) - \sinh 0$ | $\cosh x > 0$, so the square root is $\cosh x$ itself. |
| $\sinh(\ln 2) = \dfrac{2 - \frac{1}{2}}{2} = \dfrac{3}{4}$ | $e^{\ln 2} = 2$ and $e^{-\ln 2} = \frac{1}{2}$. |

## Problem 4

> Find the length of $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$ from $x = 1$ to $x = 2$.

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| $y' = \dfrac{x^2}{2} - \dfrac{1}{2x^2}$ | Differentiate term by term. |
| $1 + y'^2 = 1 + \dfrac{x^4}{4} - \dfrac{1}{2} + \dfrac{1}{4x^4} = \dfrac{x^4}{4} + \dfrac{1}{2} + \dfrac{1}{4x^4} = \left(\dfrac{x^2}{2} + \dfrac{1}{2x^2}\right)^2$ | The middle term of $y'^2$ is $-2 \cdot \frac{x^2}{2} \cdot \frac{1}{2x^2} = -\frac{1}{2}$, and adding 1 flips its sign. |
| $L = \int_1^2 \left(\dfrac{x^2}{2} + \dfrac{1}{2x^2}\right) dx = \left[\dfrac{x^3}{6} - \dfrac{1}{2x}\right]_1^2$ | The square root is gone. |
| $= \left(\dfrac{8}{6} - \dfrac{1}{4}\right) - \left(\dfrac{1}{6} - \dfrac{1}{2}\right) = \dfrac{13}{12} + \dfrac{1}{3}$ | Evaluate at both limits. |
| $\dfrac{17}{12} \approx 1.417$ | The curve was chosen so that $1 + y'^2$ is a square; most curves are not so kind. |

## Problem 5

> Find the length of the curve $x = \frac{2}{3}(y - 1)^{3/2}$ from $y = 1$ to $y = 4$.

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| $\dfrac{dx}{dy} = (y - 1)^{1/2}$ | The curve is given as $x$ in terms of $y$, so differentiate in $y$. |
| $1 + \left(\dfrac{dx}{dy}\right)^2 = 1 + (y - 1) = y$ | The square root will be $\sqrt{y}$. |

| | |
|---|---|
| $L = \int_1^4 \sqrt{y}\,dy = \frac{2}{3}\left[y^{3/2}\right]_1^4$ | The limits are values of $y$. |
| $\frac{2}{3}(8-1) = \frac{14}{3} \approx 4.667$ | Solving for $y$ and integrating in $x$ would have given the same length with a much harder integral. |

## Problem 6

Find the length of $y = \sqrt{1-x^2}$ from $x = 0$ to $x = \frac{1}{2}$, and explain the answer with a picture.

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| $y' = \frac{-x}{\sqrt{1-x^2}}$, so $1 + y'^2 = \frac{1 - x^2 + x^2}{1-x^2} = \frac{1}{1-x^2}$ | A common denominator. |
| $L = \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \arcsin\tfrac{1}{2} - \arcsin 0$ | A standard antiderivative. |
| $= \frac{\pi}{6}$ | $\sin\frac{\pi}{6} = \frac{1}{2}$. |
| $\frac{\pi}{6} \approx 0.5236$ | The curve is the unit circle, and the arc from $(0,1)$ to $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ subtends an angle of $\frac{\pi}{6}$ at the center. On a circle of radius 1, arc length is the angle. |

## Problem 7

Find the length of the parabola $y = x^2$ from $x = 0$ to $x = 1$.

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| $L = \int_0^1 \sqrt{1 + 4x^2}\,dx$ | $y' = 2x$. This one is not a perfect square. |
| $x = \frac{1}{2}\tan\theta$, $dx = \frac{1}{2}\sec^2\theta\,d\theta$, $\sqrt{1+4x^2} = \sec\theta$; $\theta$ runs from $0$ to $\arctan 2$ | Trig substitution for $\sqrt{1+(2x)^2}$. |

| | |
|---|---|
| $L = \frac{1}{2}\int_0^{\arctan 2} \sec^3\theta\, d\theta = \frac{1}{4}\Big[\sec\theta\tan\theta + \ln\lvert\sec\theta + \tan\theta\rvert\Big]_0^{\arctan 2}$ | The standard integral of $\sec^3$, done by parts. |
| At $\theta = \arctan 2$, $\tan\theta = 2$ and $\sec\theta = \sqrt{5}$ | The triangle with opposite $2$ and adjacent $1$. |
| $L = \frac{\sqrt{5}}{2} + \frac{1}{4}\ln\left(2+\sqrt{5}\right) \approx 1.4789$ | Between the chord $\sqrt{2} \approx 1.4142$ and the path along the axes, $2$. |

## Problem 8

> Turn the semicircle $y = \sqrt{r^2 - x^2}$, $-r \le x \le r$, about the $x$-axis, and find the surface area of the sphere.

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| $ds = \sqrt{1 + y'^2}\,dx = \frac{r}{\sqrt{r^2 - x^2}}\,dx$ | As in Problem 6, with radius $r$: $1 + y'^2 = \frac{r^2}{r^2 - x^2}$. |
| $2\pi y\,ds = 2\pi\sqrt{r^2 - x^2}\cdot\frac{r}{\sqrt{r^2 - x^2}}\,dx = 2\pi r\,dx$ | The radius $y$ cancels the square root. |
| $S = \int_{-r}^{r} 2\pi r\,dx = 2\pi r \cdot 2r$ | A constant integrand. |
| $S = 4\pi r^2$ | Every slice of the sphere of the same thickness $dx$ has the same area, $2\pi r\,dx$, wherever it is cut: Archimedes' hat-box theorem. |

## Problem 9

> Find the area of the surface made by turning $y = \sqrt{x}$, $0 \le x \le 2$, about the $x$-axis.

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| $y' = \frac{1}{2\sqrt{x}}$, so $\sqrt{1 + y'^2} = \sqrt{1 + \frac{1}{4x}}$ | The slant factor. |

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| $2\pi y\sqrt{1+y'^2} = 2\pi\sqrt{x}\sqrt{1+\dfrac{1}{4x}} = 2\pi\sqrt{x+\frac{1}{4}}$ | Multiply under one square root. |
| $S = 2\pi\displaystyle\int_0^2 \left(x+\frac{1}{4}\right)^{1/2} dx = \frac{4\pi}{3}\Big[\left(x+\frac{1}{4}\right)^{3/2}\Big]_0^2$ | The power rule. |
| $= \dfrac{4\pi}{3}\left(\dfrac{27}{8}-\dfrac{1}{8}\right) = \dfrac{4\pi}{3}\cdot\dfrac{13}{4}$ | $\left(\frac{9}{4}\right)^{3/2} = \frac{27}{8}$ and $\left(\frac{1}{4}\right)^{3/2} = \frac{1}{8}$. |
| $S = \dfrac{13\pi}{3} \approx 13.614$ | The surface is a paraboloid, open at the wide end. |

## Problem 10

> Find the area of the surface made by turning $y = x^2$, $0 \le x \le 1$, about the $y$-axis.

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| The distance from the point $(x, x^2)$ to the $y$-axis is $x$, so $r = x$ | About the $y$-axis, the radius is horizontal. |
| $S = \displaystyle\int_0^1 2\pi x\sqrt{1+4x^2}\,dx$ | $ds = \sqrt{1+4x^2}\,dx$ as in Problem 7. |
| $u = 1+4x^2$, $du = 8x\,dx$: $S = \dfrac{2\pi}{8}\displaystyle\int_1^5 u^{1/2}\,du = \frac{\pi}{4}\cdot\frac{2}{3}\big[u^{3/2}\big]_1^5$ | This time the $x$ in the radius makes the substitution work. |
| $S = \dfrac{\pi}{6}\left(5\sqrt{5}-1\right) \approx 5.330$ | A bowl: like the surface in Problem 9 it is a paraboloid, but here its axis is the $y$-axis, the axis of turning. |

## Where this goes wrong

### 1. Taking the square root term by term

$\sqrt{1+y'^2}$ looks as if it should simplify to $1+y'$.

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| $1+y'^2 = 1+\frac{9}{4}x$ for $y = x^{3/2}$ | Right so far: Problem 2, step 1. |
| "$\sqrt{1+\frac{9}{4}x} = 1+\frac{3}{2}\sqrt{x}$." | The habit that causes the mistake: a square root split across a sum. |
| $L = \int_0^4 \left(1+\frac{3}{2}\sqrt{x}\right)dx = 4+8 = 12$ ✗ | $\sqrt{a+b} \neq \sqrt{a}+\sqrt{b}$; squaring $1+\frac{3}{2}\sqrt{x}$ gives an extra $3\sqrt{x}$. The length is 9.073. The 12 is exactly the path along the axes, $4+8$: integrating $1+y'$ adds the horizontal distance to the vertical one. |

### 2. Using dy/dx when the integral is in y

The arc length formula is remembered with $dy/dx$ in it, whatever the variable.

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|---|---|
| $x = \frac{2}{3}(y-1)^{3/2}$ from $y=1$ to $y=4$ | Right so far: Problem 5. |
| "$\frac{dy}{dx} = \frac{1}{\sqrt{y-1}}$, so the integrand is $\sqrt{1+\frac{1}{y-1}}$." | The habit that causes the mistake: the reciprocal derivative used with $dy$. |
| $L = \int_1^4 \sqrt{\frac{y}{y-1}}\,dy = 2\sqrt{3}+\operatorname{arsinh}\sqrt{3} \approx 4.781$ ✗ | With $dy$ outside, the inside needs $\left(\frac{dx}{dy}\right)^2$: then $1+(y-1)=y$ and $L = \frac{14}{3} \approx 4.667$. The wrong answer is close enough to look right. |

### 3. Leaving out the slant factor

Surface area gets set up like a volume by shells or disks: $2\pi r$ times $dx$.

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|---|---|
| $r = \sqrt{x}$ about the $x$-axis, $0 \le x \le 2$ | Right so far: Problem 9. |
| "$S = \int_0^2 2\pi\sqrt{x}\,dx$." | The habit that causes the mistake: the band's width taken as $dx$ instead of $ds$. |

| | |
|---|---|
| $S = 2\pi \cdot \dfrac{2}{3} \cdot 2^{3/2} = \dfrac{8\sqrt{2}\,\pi}{3} \approx 11.848$ ✗ | Where the curve is steep, a band of width $dx$ is much wider along the surface. The width is $ds = \sqrt{1 + y'^2}\,dx$, and the area is $\frac{13\pi}{3} \approx 13.614$. |

## 4. Measuring the radius from the wrong axis

$2\pi y\,ds$ gets used for every surface.

| | |
|---|---|
| $ds = \sqrt{1 + 4x^2}\,dx$ for $y = x^2$ | Right so far: Problem 10, step 2. |
| "$S = \displaystyle\int_0^1 2\pi y\,ds = \int_0^1 2\pi x^2 \sqrt{1 + 4x^2}\,dx$." | The habit that causes the mistake: $y$ is the radius only about the $x$-axis. |
| $S \approx 3.810$ ✗ | That is the surface made by turning the curve about the $x$-axis, a different, narrower horn. About the $y$-axis each point travels on a circle of radius $x$, and the area is $\frac{\pi}{6}(5\sqrt{5} - 1) \approx 5.330$. |

## 5. Using the disk formula for a surface

Volumes of revolution use $\pi r^2$, and the habit carries over.

| | |
|---|---|
| $y = \sqrt{r^2 - x^2}$ and $ds = \dfrac{r}{\sqrt{r^2 - x^2}}\,dx$ | Right so far: Problem 8, step 1. |
| "$S = \displaystyle\int \pi y^2\,ds$." | The habit that causes the mistake: the area of a disk where the circumference of a band belongs. |
| $S = \displaystyle\int_{-r}^{r} \pi r \sqrt{r^2 - x^2}\,dx = \dfrac{\pi^2 r^3}{2}$ ✗ | An $r^3$ is a volume, not an area, which is the quickest check that the setup is wrong; for $r = 1$ it gives $4.935$ instead of $4\pi \approx 12.566$. A band of radius $y$ has circumference $2\pi y$, and $\int 2\pi y\,ds = 4\pi r^2$. |

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